Annex C[7]

(informative)

An overview of the Allan variance method of IFOG noise analysis

C.1 Allan variance background

Allan variance is a time domain analysis technique originally developed to study the frequency stability of oscillators [C1].[8] It can be used to determine the character of the underlying random processes that give rise to the data noise. As such, it helps identify the source of a given noise term in the data. The source may be inherent in the instrument, but in the absence of a plausible mechanism within the instrument its origin should be sought in the test set up. The Allan variance adopted in this standard may be used as a stand-alone method of data analysis or to complement any of the frequency domain analysis techniques. It should be mentioned that the technique can be applied to the noise study of any instrument. Its value, however, depends upon the degree of understanding of the physics of the instrument. Following is an overview of the Allan variance and its adaptation to the noise properties of IFOGs, similar to that described in [C6] for ring laser gyros.

In the Allan variance method of data analysis, the uncertainty in the data is assumed to be generated by noise sources of specific character. The magnitude of each noise source covariance is then estimated from the data. The definition of the Allan variance and a discussion of its use in frequency and time metrology is presented in [C1] and [C7].

In this annex, Allan's definition and results are related to five basic gyro noise terms and are expressed in a notation appropriate for gyro data reduction. The five basic noise terms are angle random walk, rate random walk, bias instability, quantization noise, and rate ramp.

Consider N samples of gyro data[9] with a sample time of τ_o. Form data clusters of lengths τ_o, $2\tau_o$, ..., $k\tau_o$ ($k<N/2$) and obtain averages of the sum of the data points contained in each cluster over the length of that cluster. The Allan variance is defined as a function of cluster time.

To be specific, the Allan variance can be defined either in terms of the output rate, $\Omega(t)$, or the output angle

$$\theta(t) = \int^{t} \Omega(t')dt'$$

The lower integration limit is not specified as only angle differences are employed in the definitions. Angle measurements are made at discrete times given by $t = k\tau_o$, $k = 1, 2, 3, ..., N$. Accordingly, the notation is simplified by writing $\Theta_k = \Theta(k\tau_o)$.

The average rate between times t_k and $t_k + \tau$ is given by:

$$\bar{\Omega}_k(\tau) = \frac{\theta_{k+m} - \theta_k}{\tau}$$

where

$\tau = m\tau_o$

[7]This annex is adapted from Annex C in IEEE Std 647-1995, IEEE Standard Specification Format Guide and Test Procedure for Single-Axis Laser Gyros.

[8]The numbers in brackets preceded by the letter C correspond to those of the bibliography in C.4.

[9]Sometimes referred to as time series or data streams.

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The Allan variance[10] is defined as:

$$\sigma^2(\tau) = \frac{1}{2}\langle(\bar{\Omega}_{k+m} - \bar{\Omega}_k)^2\rangle$$

$$= \frac{1}{2\tau^2}\langle(\theta_{k+2m} - 2\theta_{k+m} + \theta_k)^2\rangle$$

where

$\langle\ \rangle$ is the ensemble average

The Allan variance is estimated as follows:

$$\sigma^2(\tau) = \frac{1}{2\tau^2(N-2m)}\sum_{k=1}^{N-2m}(\theta_{k+2m} - 2\theta_{k+m} + \theta_k)^2$$

The Allan variance obtained by performing the prescribed operations, is related to the PSD of the noise terms in the original data set. The relationship between Allan variance and the two-sided PSD[11], $S_\Omega(f)$ is given by:

$$\sigma^2(\tau) = 4\int_0^\infty S_\Omega(f)\frac{\sin^4(\pi f\tau)}{(\pi f\tau)^2}df \tag{C.1}$$

Equation (C.1) is the key result that will be used to calculate the Allan variance from the rate noise PSD. An interpretation is that the Allan variance is proportional to the total noise power of the gyro rate output when passed through a filter with the transfer function of the form $\sin^4(x)/(x)^2$. This particular transfer function is the result of the method used to create and operate on the clusters.

It is seen from Equation (C.1) and the above interpretation that the filter bandpass depends on τ. This suggests that different types of random processes can be examined by adjusting the filter bandpass, namely by varying τ. Thus, the Allan variance provides a means of identifying and quantifying various noise terms that exist in the data. It is normally plotted as the square root of the Allan variance versus τ, $[\sigma(\tau)]$, on a log-log plot.

Subclauses C.1.1 through C.1.7 show the application of Equation (C.1) to a number of noise terms that are either known to exist in the IFOG or otherwise influence its data. Detailed derivations are given in [C6]. The physical origin of each noise source term will be discussed.

[10] Frequently the term Allan variance is also used to refer to its square root, $\sigma(\tau)$.

[11] Unless specifically stated, the term PSD is Annex C refers to the two-sided PSD. Note that $S_\Omega(f)$ is the PSD of stationary random processes. For nonstationary processes, such as flicker noise, the time average PSD should be used.

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C.1.1 Angle random walk

The main source for this error is spontaneous emission of photons. This component of the IFOG angle random walk is caused by the spontaneously emitted photons that are always present in the source output. The angle random walk due to spontaneously emitted photons is called the quantum limit [C4].

Other high frequency noise terms that have correlation time much shorter than the sample time, can also contribute to the gyro angle random walk. However, most of these sources can be eliminated by design. These noise terms are all characterized by a white noise spectrum on the gyro rate output.

The associated rate noise PSD is represented by:

$$S_{\Omega}(f) = N^2 \tag{C.2}$$

where

N is the angle random walk coefficient[12]

Substitution of Equation (C.2) in Equation (C.1) and performing the integration yields:

$$\sigma^2(\tau) = \frac{N^2}{\tau} \tag{C.3}$$

As shown in Figure C.1, Equation (C.3) indicates that a log-log plot of $\sigma(\tau)$ versus τ has a slope of $-1/2$. Furthermore, the numerical value of N can be obtained directly by reading the slope line at $\tau = 1$.

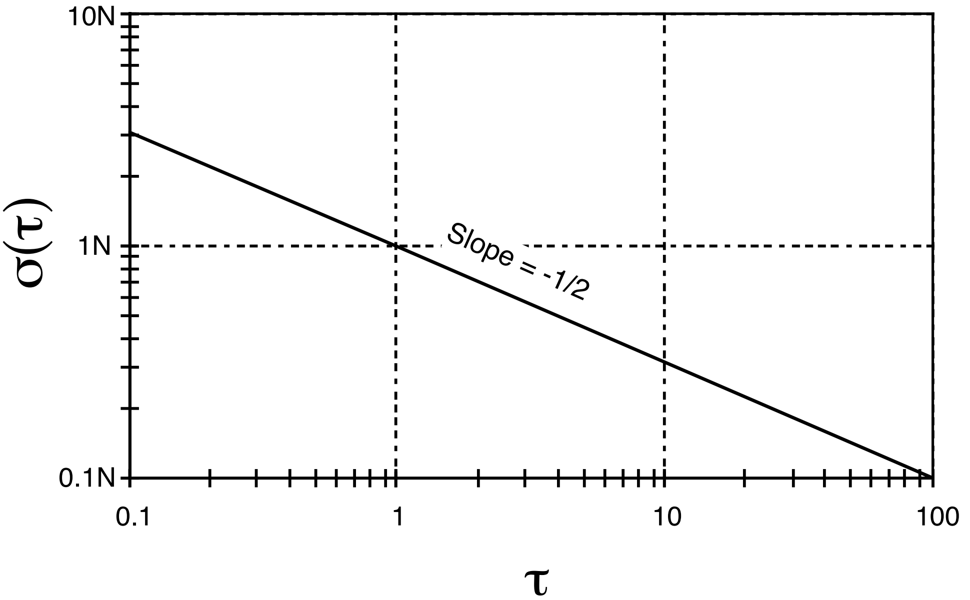

Figure C.1—$\sigma(\tau)$ Plot for angle random walk

[12]The zero slop portion of the rate PSD in $(°/h)^2/Hz$ represents angle random walk. The relationship between angle random walk coefficient N and the PSD is

$$N(°/\sqrt{h}) = \frac{1}{60}\sqrt{PSD\left[\left(\frac{°}{h}\right)^2/Hz\right]}$$

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C.1.2 Bias instability

The origin of this noise is the electronics, or other components susceptible to random flickering [C5]. Because of its low-frequency nature it shows up as the bias fluctuations in the data. The rate PSD associated with this noise is:

$$S_{\Omega}(f) = \begin{cases} \left(\frac{B^2}{2\pi}\right)\frac{1}{f} & f \le f_0 \\ 0 & f > f_0 \end{cases} \tag{C.4}$$

where

B is the bias instability coefficient
f_0 is the cutoff frequency

Substitution of Equation (C.4) in Equation (C.1) and performing the integration yields:

$$\sigma^2(\tau) = \frac{2B^2}{\pi}\left[\ln 2 - \frac{\sin^3 x}{2x^2}(\sin x + 4x\cos x) + Ci(2x) - Ci(4x)\right] \tag{C.5}$$

where

x is $\pi f_0 \tau$
Ci is the cosine-integral function [C2]

Figure C.2 represents a log-log plot of Equation (C.5) that shows that the Allan variance for bias instability reaches a plateau for τ much longer than the inverse cut off frequency. Thus, the flat region of the plot can be examined to estimate the limit of the bias instability as well as the cutoff frequency of the underlying flicker noise.

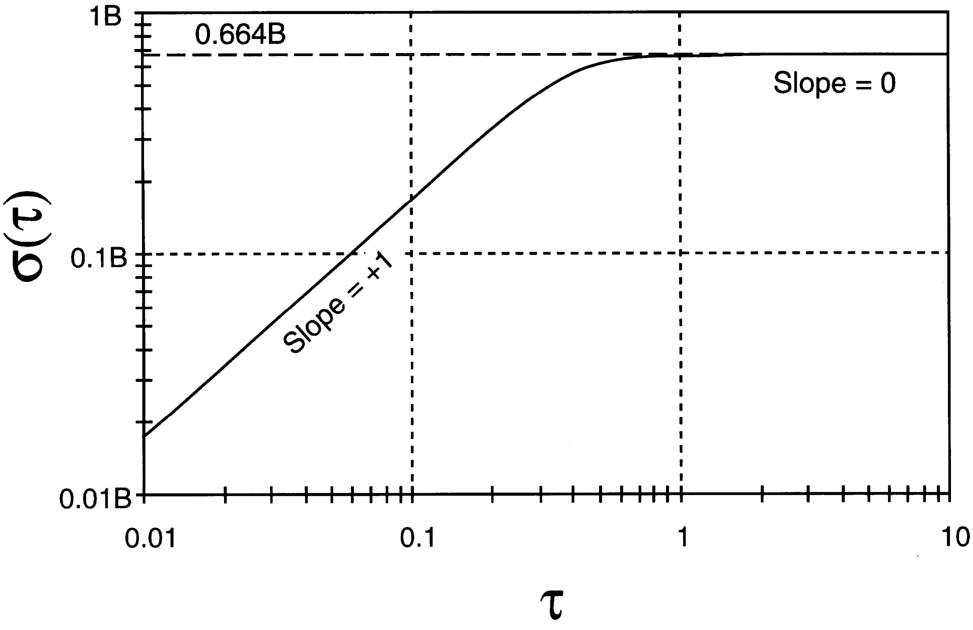

Figure C.2—$\sigma(\tau)$ Plot for bias instability (for $f_0 = 1$)

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C.1.3 Rate random walk

This is a random process of uncertain origin, possibly a limiting case of an exponentially correlated noise with a very long correlation time, as discussed in Clause 3.

The rate PSD associated with this noise is:

$$S_\Omega(f) = \left(\frac{K}{2\pi}\right)^2 \frac{1}{f^2} \tag{C.6}$$

where

K is the rate random walk coefficient

Substitution of Equation (C.6) in Equation (C.1) and performing the integration yields:

$$\sigma^2(\tau) = \frac{K^2\tau}{3} \tag{C.7}$$

This indicates that rate random walk is represented by a slope of +1/2 on a log-log plot of $\sigma(\tau)$ versus τ, as shown in Figure C.3. The magnitude of this noise can be read off the slope line at $\tau = 3$.

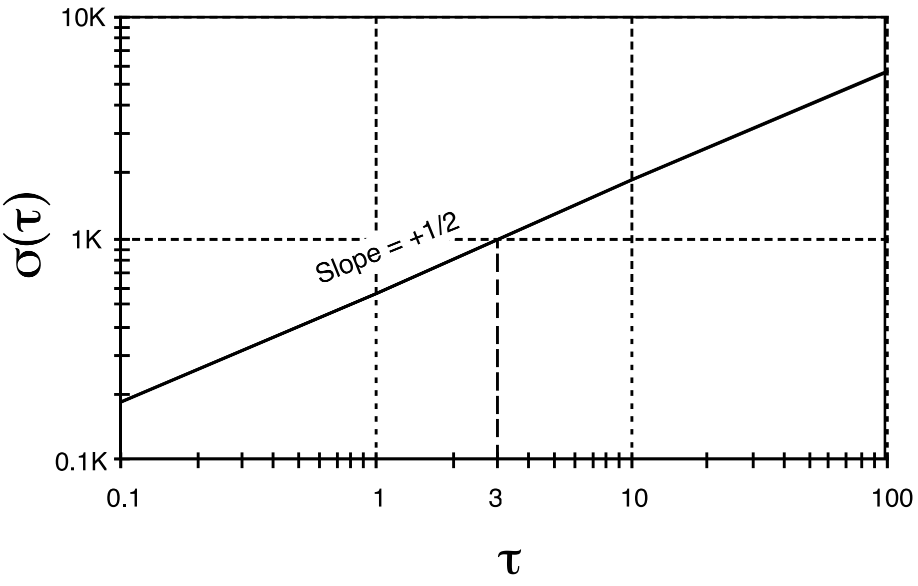

Figure C.3—$\sigma(\tau)$ Plot for rate random walk

C.1.4 Rate ramp

For long, but finite time intervals this is more of a deterministic error rather than a random noise. Its presence in the data may indicate a very slow monotonic change of the IFOG source intensity persisting over a long period of time. It could also be due to a very small acceleration of the platform in the same direction and persisting over a long period of time. It appears as a genuine input to the IFOG given by:

$$\Omega = Rt \tag{C.8}$$

where

R is the rate ramp coefficient

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By forming and operating on the clusters of data containing an input given by Equation (C.8), we obtain:

$$\sigma^2(\tau) = \frac{R^2\tau^2}{2} \tag{C.9}$$

This indicates that the rate ramp noise has a slope of +1 in the log-log plot of $\sigma(\tau)$ versus τ, as shown in Figure C.4. The magnitude of rate ramp R can be obtained from the slope line at $\tau = \sqrt{2}$.

The rate PSD associated with this noise is:

$$S_\Omega(f) = \frac{R^2}{(2\pi f)^3} \tag{C.10}$$

The user should be aware that there may be a flicker acceleration noise with $1/f^3$ PSD that leads to the same Allan variance τ dependence. See Annex B for a discussion.

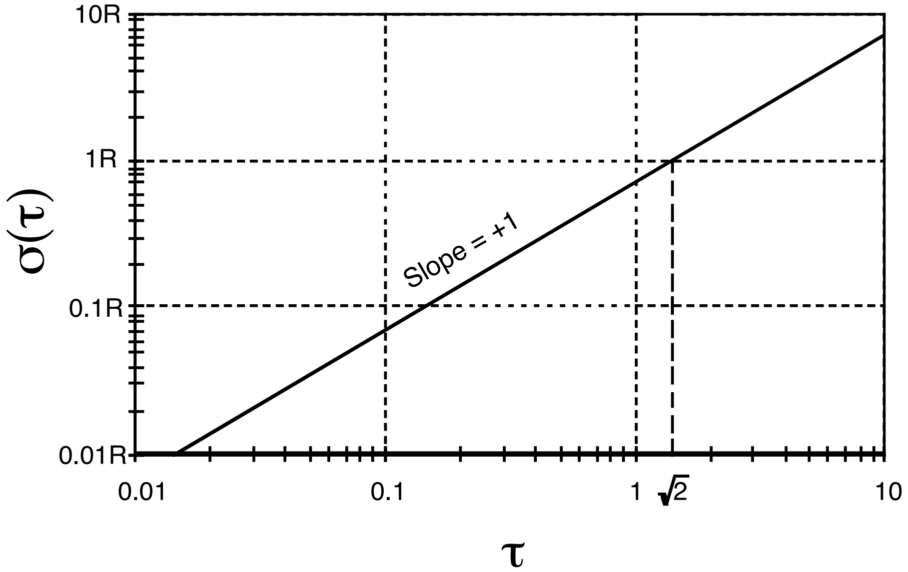

Figure C.4—$\sigma(\tau)$ plot for rate ramp

C.1.5 Quantization noise

This noise is strictly due to the digital nature of the IFOG output. The readout electronics registers a count only when the gyro phase changes by a predetermined amount, e.g., $2\pi/2^n$, where $n = 0, 1, 2, ...$

The angle PSD for such a process, given in [C8] is:

$$S_\theta(f) = \begin{cases} \tau_0 Q^2 \left(\dfrac{\sin^2(\pi f \tau_0)}{(\pi f \tau_0)^2} \right) \\ \approx \tau_0 Q^2 \qquad f < \dfrac{1}{2\tau_0} \end{cases} \tag{C.11}$$

where

Q is the quantization noise coefficient

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The theoretical limit for Q is equal to $S/\sqrt{12}$ where S is the gyro scale factor, for tests with fixed and uniform sampling times. The rate PSD is related to the angle PSD through the equation:

$$S_{\Omega}(2\pi f) = (2\pi f)^2 S_{\theta}(2\pi f) \quad \text{(C.12)}$$

and is

$$S_{\Omega}(f) = \begin{cases} \dfrac{4Q^2}{\tau_0}\sin^2(\pi f \tau_0) \\ \approx (2\pi f)^2 \tau_0 Q^2 & f < \dfrac{1}{2\tau_0} \end{cases} \quad \text{(C.13)}$$

Substitution of Equation (C.13) in Equation (C.1) and performing the integration yields:

$$\sigma^2(\tau) = \frac{3Q^2}{\tau^2} \quad \text{(C.14)}$$

This indicates that the quantization noise is represented by a slope of –1 in a log-log plot of $\sigma(\tau)$ versus τ, as shown in Figure C.5. The magnitude of this noise can be read off the slope line at $\tau = \sqrt{3}$.

The user should be aware that there are other noise terms with different spectral characteristics, such as flicker angle noise and white angle noise, that lead to the same Allan variance τ dependence. See Annex B for a discussion of these noise terms.

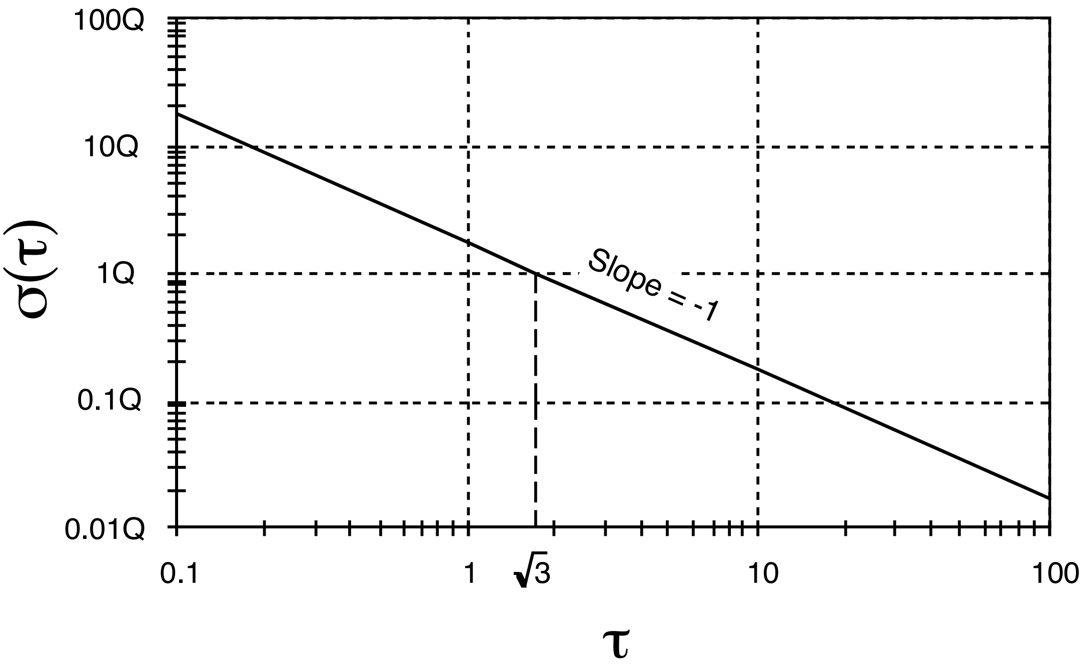

Figure C.5—$\sigma(\tau)$ Plot for quantization noise

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C.1.6 Other noise terms

C.1.6.1 Exponentially correlated (Markov) noise

This noise is characterized by an exponential decaying function with a finite correlation time.

The rate PSD for such a process:

$$S_\Omega(f) = \frac{(q_c T_c)^2}{1 + (2\pi f T_c)^2} \tag{C.15}$$

where

q_c	is the noise amplitude
T_c	is the correlation time

Substitution of Equation (C.15) in Equation (C.1) and performing the integration yields:

$$\sigma^2(\tau) = \frac{(q_c T_c)^2}{\tau}\left[1 - \frac{T_c}{2\tau}\left(3 - 4e^{-\frac{\tau}{T_c}} + e^{-\frac{2\tau}{T_c}}\right)\right] \tag{C.16}$$

Figure C.6 shows a log-log plot of Equation (C.16). It is instructive to examine various limits of this equation. For τ much longer than the correlation time, it is found that:

$$\sigma^2(\tau) \Rightarrow \frac{(q_c T_c)^2}{\tau} \qquad \tau \gg T_c \tag{C.17}$$

which is the Allan variance for angle random walk where $N = q_c T_c$ is the angle random walk coefficient. For τ much smaller than the correlation time, Equation (C.16) reduces to:

$$\sigma^2(\tau) \Rightarrow \frac{q_c^{\ 2}}{3}\tau \qquad \tau \ll T_c \tag{C.18}$$

which is the Allan variance for rate random walk.

C.1.6.2 Sinusoidal noise

The PSD of this noise is characterized by one or more distinct frequencies. A low-frequency source could be the slow motion of the test platform due to periodic environmental changes. A representation of the PSD of this noise containing a single frequency is given as:

$$S_\Omega(f) = \frac{1}{2}\Omega_0^2[\delta(f - f_0) + \delta(f + f_0)] \tag{C.19}$$

where

Ω_0	is the amplitude
f_0	is the frequency
$\delta(x)$	is the Dirac delta function

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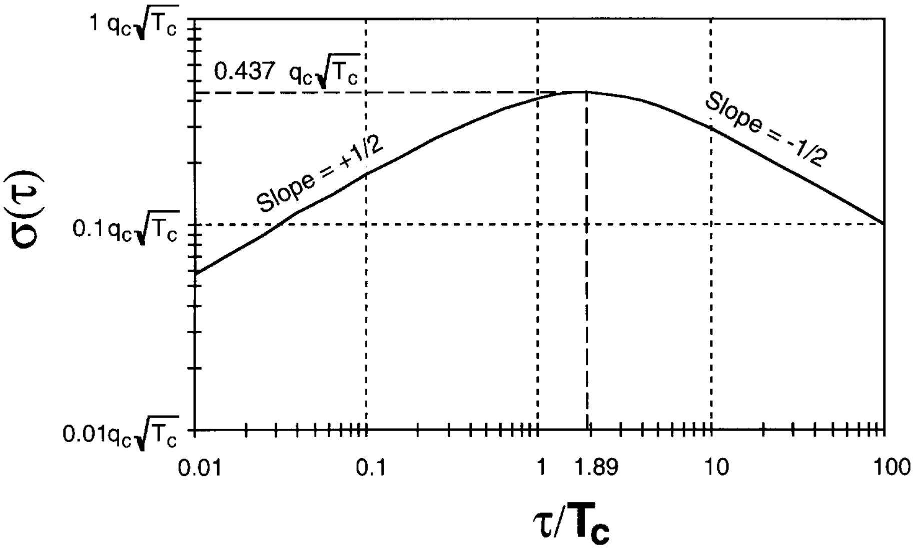

Figure C.6—σ(τ) Plot for correlated noise

Multiple frequency sinusoidal errors can be similarly represented by a sum of terms such as Equation (C.19) at their respective frequencies and amplitudes. Substitution of Equation (C.19) in Equation (C.1) and performing the integration yields:

$$\sigma^2(\tau) = \Omega_0^2\left(\frac{\sin^2 \pi f_0 \tau}{\pi f_0 \tau}\right)^2 \tag{C.20}$$

Figure C.7 shows a log-log plot of Equation (C.20). Identification and estimation of this noise in IFOG data requires the observation of several peaks. As is seen however, the amplitudes of consecutive peaks fall off rapidly and may be masked by higher order peaks of other frequencies making observation difficult.

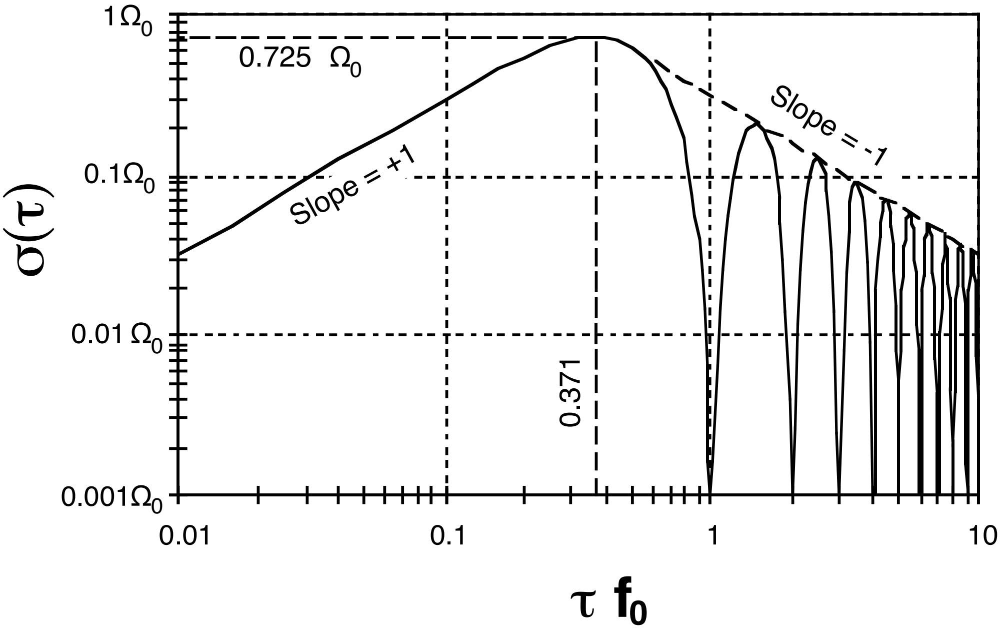

Figure C.7—σ(τ) Plot for sinusoidal error

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C.1.7 Combined effects of all processes

In general, any number of the random processes discussed above (as well as others) can be present in the data. Thus, a typical Allan variance plot looks like the one shown in Figure C.8. Experience shows that in most cases, different noise terms appear in different regions of τ. This allows easy identification of various random processes that exist in the data. If it can be assumed that the existing random processes are all statistically independent then it can be shown that the Allan variance at any given τ is the sum of Allan variances due to the individual random processes at the same τ. In other words,

$$\sigma^2_{tot}(\tau) = \sigma^2_{ARW}(\tau) + \sigma^2_{quant}(\tau) + \sigma^2_{BiasInst}(\tau) + \ldots \quad \text{(C.21)}$$

Thus estimating the amplitude of a given random noise in any region of τ requires a knowledge of the amplitudes of the other random noises in the same region.

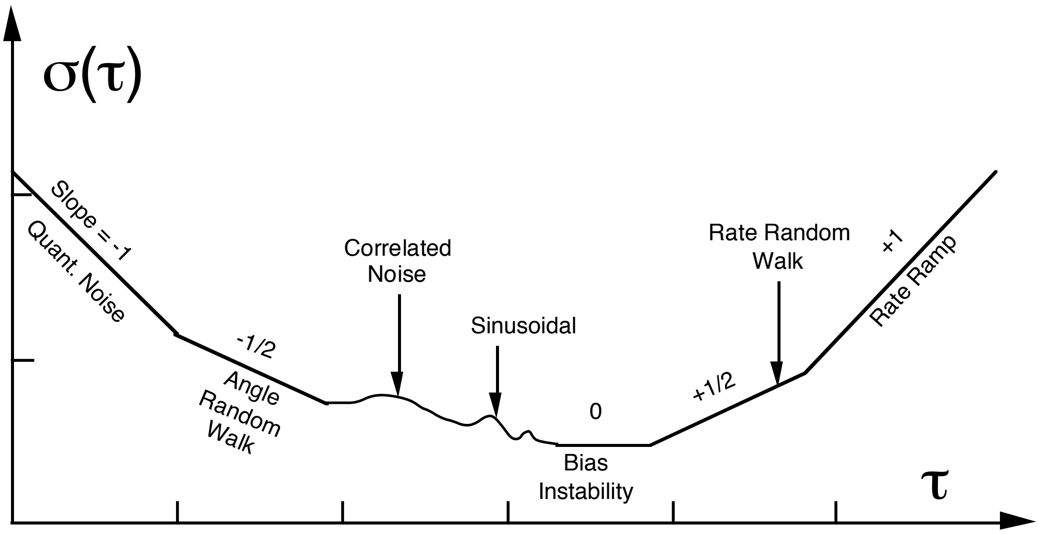

Figure C.8—$\sigma(\tau)$ Sample plot of Allan variance analysis results

C.2 Estimation accuracy and test design

A finite number of clusters can be generated from any finite set of data. Allan variance of any noise term is estimated using the total number of clusters of a given length that can be created. Estimation accuracy of the Allan variance for a given τ, on the other hand, depends on the number of independent clusters within the data set.

It can be shown that the percentage error, σ, in estimating $\sigma(\tau)$ when using clusters containing K data points from a data set of N points is given by:

$$\sigma = \frac{1}{\sqrt{2\left(\frac{N}{K} - 1\right)}} \quad \text{(C.22)}$$

Equation (C.22) shows that the estimation errors in the regions of short (long) τ are small (large) as the number of independent clusters in these regions is large (small). In fact, this equation can be used to design a test to observe a particular noise of certain characteristics to within a given accuracy. For example, to verify the existence of a random process with a characteristic time of 24 h in the data to within an error of 25%. We first set $\sigma = 0.25$ in Equation (C.22) and obtain:

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$$K_{max} = \frac{N}{9} \quad \text{(C.23)}$$

Since the suspected characteristic time is 24 h, clusters of the same length are created. Thus the total test length needed for such a test is 24 × 9 = 216 h.

C.3 Tabulation of some variance analyses

A summary comparison of some variance analyses for noise processes is made in Table C.1. This table presents only a sample of analyses available, and is not meant to be a survey of all analyses. The polynomial variance terms in the left hand column are identified using gyro terminology. The individual terms relating to each author's publication are given with the same symbology as contained in that author's publication, including the definitions of symbols. For ease in recognition of similarities, the coefficients of interest are shown as the first symbol in each polynomial expression. For example, the variance coefficient for the rate random walk term in the third column of Table C.1, is K^2.

Table C.1—Summary comparison of publishing variance analyses for noise processes

	Allan [C1] (rate domain)	**This standard (rate domain)**	**Sargent, Wyman [C3] (angle domain)**	**Tehrani [C6] (rate domain)**
Terms in variance expression for noise processes	τ = sampling time = $m\tau_o$ where τ_o = sample time of original measurements	$1/T_0$ = data sample rate n = 1, 2, 3,...	γ = time interval separating raw data points ($2L$+1) = number of data points combined into an average data point τ = time length of data span = $(2L+1)\gamma n$ n = 1, 2, 3,...	$Q = qT_c$ T_c = correlation time T = cluster time v_o = cutoff frequency for $1/v$ rate noise
Rate ramp	Not addressed	$R^2\left[\frac{(nT_o)^2}{2}\right]$	$R^2[\tau^4]$	$R^2[T^2]$
Rate random walk	$h_{-2}\left[\frac{(2\pi)^2\tau}{6}\right]$	$K^2\left[\frac{nT_o}{3}\right]$	$K^2\left[\frac{2\tau^3}{3} - \frac{2L(L+1)\gamma^2\tau}{3} + \frac{L(L+1)(12L^2+12L+1)\gamma^3}{15(2L+1)}\right]$	$q^2\left[\frac{2T}{3}\right]$ for $T \le T_0$
Bias instability	$h_{-1}[2\ln 2]$	$B^2\left[\frac{2}{\pi}\ln 2\right]$	Analysis considers only time-dependent rate terms	$B^2\left[\frac{4}{\pi}\ln 2\right]$ for $T \gg \frac{1}{v_o}$
Angle random walk	$h_{-1}\left[\frac{1}{2\tau}\right]$	$N^2\left[\frac{1}{nT_o}\right]$	$\sigma^2\left[2\tau - \frac{4\gamma L(L+1)}{2L+1}\right]$	$Q^2\left[\frac{2}{T}\right]$
Quantization noise	Not addressed	$Q^2\left[\frac{3}{(nT_o)^2}\right]$	$\Phi^2\left[\frac{6}{2L+1}\right]$	Not addressed

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C.4 Bibliography for Annex C

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[13]IEEE Std 1139-1988 has been withdrawn; however, copies can be obtained from Global Engineering, 15 Inverness Way East, Englewood, CO 80112-5704, USA, tel. (303) 792-2181.

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