GNSS/INS紧组合算法原理

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热身：条件期望与方差

- 最小二乘估计
  - 期望：$\begin{pmatrix} x_a \\ x_b \end{pmatrix}$
  - 协因数阵：$\begin{pmatrix} Q_a & Q_{ab} \\ Q_{ba} & Q_b \end{pmatrix}$
- 条件期望

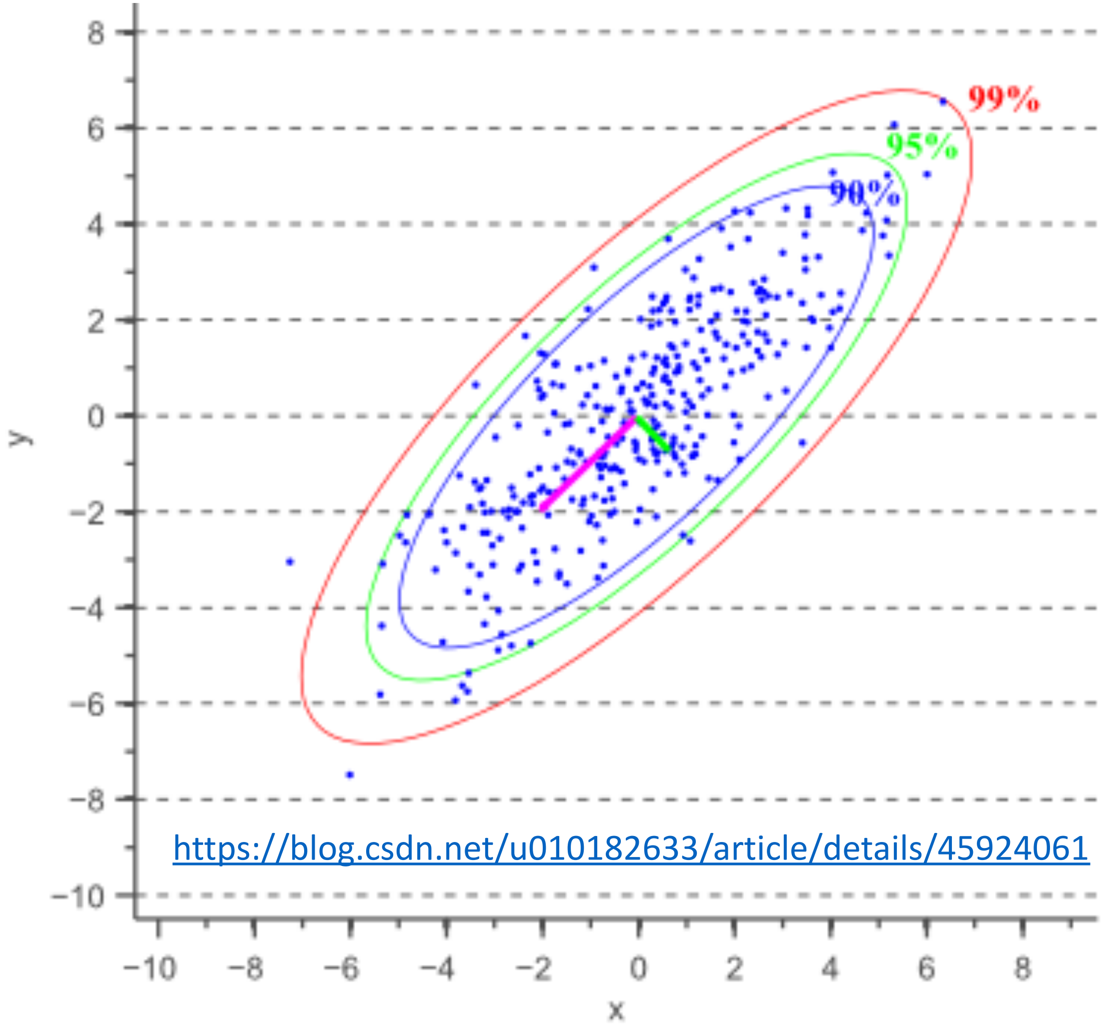
$$\begin{pmatrix} x_a \\ x_{b|a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Q_{ba}Q_a^{-1} & 1 \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix} = \begin{pmatrix} x_a \\ x_b - Q_{ba}Q_a^{-1}x_a \end{pmatrix}$$

- 条件方差

$$Q' = \begin{pmatrix} 1 & 0 \\ -Q_{ba}Q_a^{-1} & 1 \end{pmatrix} \begin{pmatrix} Q_a & Q_{ab} \\ Q_{ba} & Q_b \end{pmatrix} \begin{pmatrix} 1 & -Q_a^{-1}Q_{ab} \\ 0 & 1 \end{pmatrix}$$

$$Q' = \begin{pmatrix} Q_a & 0 \\ 0 & Q_b - Q_{ba}Q_a^{-1}Q_{ab} \end{pmatrix}$$

- 参数 $x_{b|a}$估值协因数阵 $Q_b - Q_{ba}Q_a^{-1}Q_{ab} \leq Q_b$
  - 参数$x_{b|a}$估计精度高于x_b



https://blog.csdn.net/u010182633/article/details/45924061

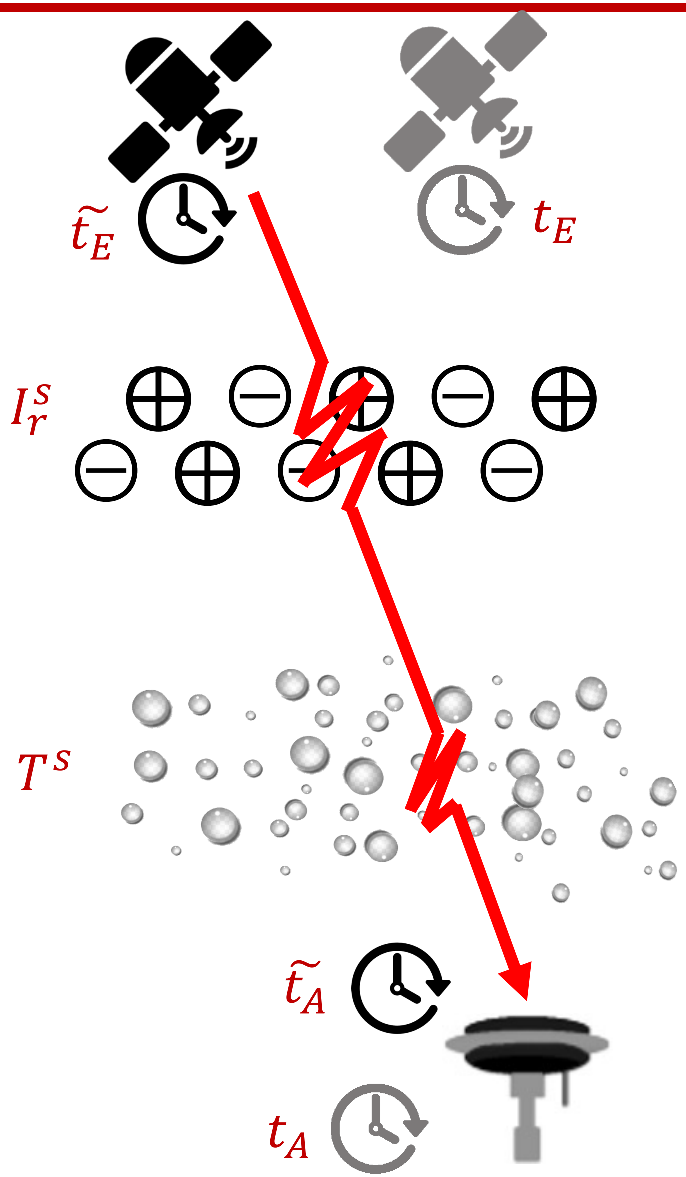
GNSS伪距观测方程

- $t_A = t_E + \tau(t_A)$
- $c \cdot \tau(t_A) = \rho_r^s(t_A) + d\rho + T^s + \frac{40.3}{f^2} I_r^s$
- $\tilde{t}_A = t_A + \frac{t_r(t_A) + b_r}{c}$
- $\tilde{t}_E = t_E + \frac{\left(t^s(t_E) + \delta t^{rel}(t_E)\right) + b^s}{c}$
- Summarizing these terms leads to

$$P_r^s(t_A) = c(\tilde{t}_A - \tilde{t}_E) + d\rho + T^s + \frac{40.3}{f^2} I_r^s$$

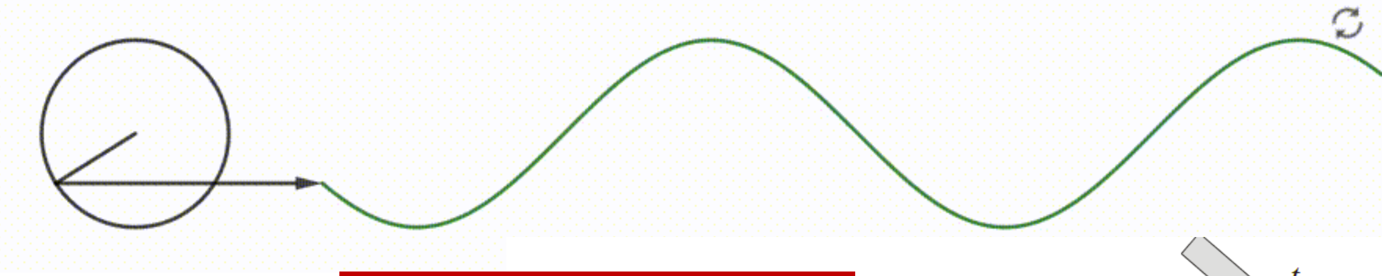
$$P_r^s(t_A) = \rho_r^s(t_A) + d\rho - \left(t^s + \delta t^{rel}\right)(t_E) + t_r(t_A) - b^s + b_r + T^s + \frac{40.3}{f^2} I_r^s + \varepsilon_P$$

τ:	is the signal propagation time (m)
ρ_r^s:	is the geometric range (m)
$d\rho$:	including PCO/PCV, Solid earth tides, Polar tides and Ocean loading etc.
T^s:	is the slant tropospheric delay (m)
I_r^s:	is the slant ionosphere delay (TECU)
b_r^s:	is the code bias (TGD) for pesudorange (m)
P_r^s:	is the measured pseudorange (m)



GNSS相位观测方程

$P_r^s = \rho_r^s - \left(t^s + \delta t^{rel}\right) + t_r - b^s + b_r + T^s + \frac{40.3}{f^2} I_r^s$



Continuous tracking

- 相位观测
  - 首次观测: $\varphi_0 = Fr(\varphi)_0$
  - 后续历元: $\varphi_i = Int(\varphi)_i + Fr(\varphi)_i$
- 相位测量距离

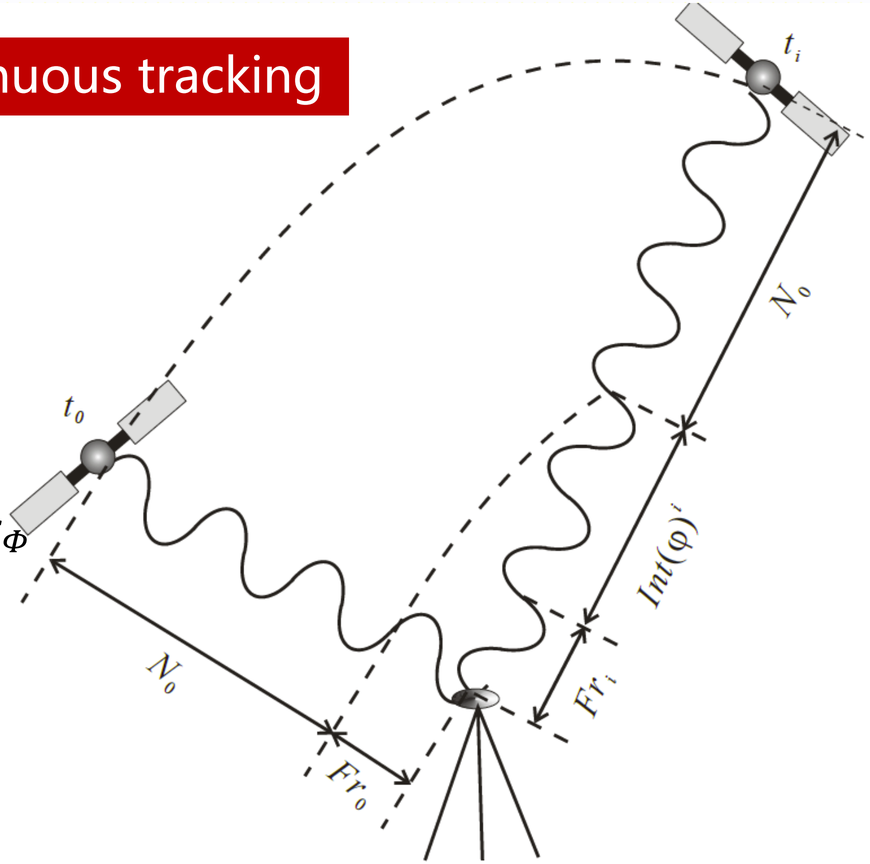
$$\Phi = \lambda_f\left(Int(\varphi)_i + Fr(\varphi)_i\right) = \rho_r^s - \lambda_f n_0$$

  - 整周计数: $\mathrm{Int}(\varphi)$
  - 整周模糊度: n_0
- 考虑各项误差，相位观测方程

$$\Phi = \rho_r^s - \left(t^s + \delta t^{rel}\right) + t_r - d^s + d_r + T^s - \frac{40.3}{f^2} I_r^s - \lambda_f\left(n_{r,f}^s + \varphi_r^s\right) + \varepsilon_\Phi$$

$$\Phi = \rho_r^s - \left(t^s + \delta t^{rel}\right) + t_r + T^s - \frac{40.3}{f^2} I_r^s - \lambda_f\left(N_{r,f}^s + \varphi_r^s\right) + \varepsilon_\Phi$$

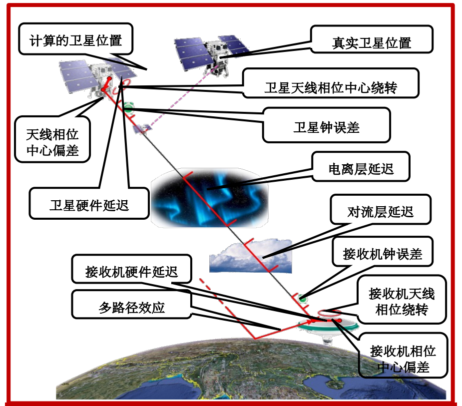
  - 相位缠绕： φ_r^s
  - 浮点模糊度: $N_{r,f}^s = n_{r,f}^s + d^s - d_r$
  - 相位偏差（UPD/FCB）： d^s， d_r



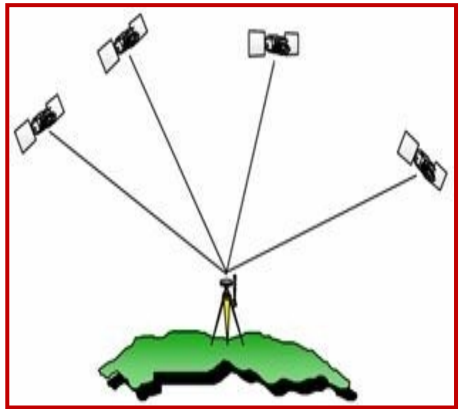
标准单点定位SPP与精密单点定位PPP

- 标准单点定位（standard point positioning, SPP）是指利用广播星历卫星轨道和钟差产品，在考虑部分误差改正后，采用合理的参数估计策略（一般为最小二乘），利用单台GNSS接收机伪距观测值实现全球米级绝对定位的技术，是GNSS标准定位服务模式
- 精密单点定位（precise point positioning, PPP）是指利用外部组织（如IGS或商业公司）提供的精密卫星轨道和钟差产品，在综合考虑各项误差精确改正的基础上，采用合理的参数估计策略（如最小二乘或Kalman滤波等），利用单台GNSS接收机伪距和相位观测值实现全球mm–dm级绝对定位的技术，PPP有发展为系统内置服务的趋势

精密卫星轨道钟差



各项误差精确改正


单台GNSS接收机


全球绝对定位

GNSS误差源

$$\begin{cases} P_r^s = \rho_r^s - (t^s + \delta t^{rel}) + t_r - b^s + b_r + T^s + \frac{40.3}{f^2} I_r^s + \varepsilon_P \\ \Phi = \rho_r^s - (t^s + \delta t^{rel}) + t_r + T^s - \frac{40.3}{f^2} I_r^s - \lambda_f (N_{r,f}^s + \varphi_r^s) + \varepsilon_\Phi \end{cases}$$

- 外部数据源
  - IGS公布（卫星产品等）
- 模型改正
  - IERS Convention 2010（参考框架等）
  - 数值/经验模型（大气延迟等）
- 参数估计
  - 无法完全模型化误差
  - 未模型化误差具备一定规律
  - 误差 VS 信号

误差源	量级 [m]
Satellite orbit	~1
Satellite clock	~1
Satellite phase center	~1
Satellite code bias/Satellite phase bias	~1
Phase wind up	~0.1
Ionospheric delay	~10
Tropospheric delay	~3
Earth rotation	~30
Relativistic effect	~5
Multi path	/
Receiver clock	/
Receiver phase center	~1
Receiver code bias/Receiver phase bias	~1
Solid earth tide/Polar tides/Ocean loading	~0.1

精密卫星轨道钟差

- IGS产品
  - ftp://igs.ign.fr/pub/igs/products
  - ftp://cddis.gsfc.nasa.gov/pub/gps/products
- MGEX产品
  - ftp://igs.ign.fr/pub/igs/products/mgex
  - ftp://cddis.gsfc.nasa.gov/pub/gps/products/mgex
- BDS-3产品
  - ftp://cddis.gsfc.nasa.gov/pub/gnss/products/mgex
  - ftp://igs.gnsswhu.cn/pub/gnss/products/mgex

Table 1 Overview of the IGS and MGEX ACs and precise products

Institution	Prefix	System	Orbit/clock	Remarks
IGS				
CODE	*cod*	GR	15 min/5 s	–
NRCan	*emr*	G	15 min/30 s	–
ESA/ESOC	*esa*	GR	15 min/30 s	–
GFZ	*gfz*	GR	15 min/30 s	–
CNES/CLS	*grg*	GR	15 min/30 s	–
IGS	*igs*	G	15 min/30 s	Official combined products
JPL	*jpl*	G	15 min/30 s	–
MIT	*mit*	G	15 min/30 s	–
NGS	*ngs*	G	15 min/15 min	Excluded
SIO	*sio*	G	15 min/15 min	Excluded
MGEX				
CODE	*com*	GRCEJ	5 min/30 s	–
GFZ	*gbm*	GRCEJ	5 min/30 s	–
CNES/CLS	*grm*	GRE	15 min/30 s	–
JAXA	*jax*	GRJ	5 min/30 s	–
SHAO	*sha*	GRCE	15 min/5 min	Excluded
TUM	*tum*	CEJ	5 min/5 min	Excluded
WHU	*wum*	GRCEJ	15 min/30 s	–

实时精密卫星轨道钟差

- 武汉大学实时产品
  - C/G/R/E四系统实时轨道、钟差
  - CLK15：质心（CoM）
  - CLK16：相位中心（APC）
- 法国宇航局实时产品
  - C/G/R/E四系统实时轨道、钟差、相位偏差
  - 全球电离层延迟
  - CLK90/CLK92：质心（CoM）
  - CLK91/CLK93：相位中心（APC）
- 实时产品获取
  - NTRIP协议
  - BNC软件ftp://igs.bkg.bund.de/NTRIP/software/

mountpoint ▲	identifier	misc
CLK00	BRDC_CoM_ITRF	BKG
CLK01	BRDC_CoM_ITRF	BKG
CLK10	BRDC_APC_ITRF	BKG
CLK10_DREF91	BRDC_APC_ITRF	BKG
CLK11	BRDC_APC_ITRF	BKG
CLK11_DREF91	BRDC_APC_ITRF	BKG
CLK15	BRDC_CoM_ITRF	WUHAN
CLK16	BRDC_APC_ITRF	WUHAN
CLK21	BRDC_CoM_ITRF	gnss.gsoc.dlr.de:2101/CLKC0_DEU1(1)
CLK22	BRDC_APC_ITRF	NRCan
CLK24	BRDC_CoM_ITRF	IGS Combination
CLK25	BRDC_APC_ITRF	IGS Combination
CLK30	BRDC_CoM_ITRF	IGS Single-Epoch Combination
CLK31	BRDC_APC_ITRF	IGS Single-Epoch Combination
CLK50	BRDC_CoM_ITRF	ESA/ESOC
CLK51	BRDC_APC_ITRF	ESA/ESOC
CLK52	BRDC_CoM_ITRF	ESA/ESOC2
CLK53	BRDC_APC_ITRF	ESA/ESOC2
CLK90	BRDC_CoM_ITRF	CNES/ORB
CLK91	BRDC_APC_ITRF	CNES/ORB
CLK92	BRDC_CoM_ITRF	Phase CNES/ORB
CLK93	BRDC_APC_ITRF	Phase CNES/ORB

GNSS非差非组合PPP观测方程

$$\begin{cases} P_f^s = \rho_r^s + t_r + T^s + \frac{40.3}{f^2} I_r^s - b^{s,f} + b_{r,f} + \varepsilon_P \\ \Phi_f^s = \rho_r^s + t_r + T^s - \frac{40.3}{f^2} I_r^s - \lambda_f N_{r,f}^s + \varepsilon_\Phi \end{cases}$$

P_f^s:	is the measured pseudorange on frequency f (m)
Φ_f^s:	is the measured carrier phase on frequency f (m)
ρ_r^s:	is the true geometric range, $\rho_r^s = \lvert r^s - r_r \rvert$ (m)
t_r:	is the clock error for pesudorange and carrier phase (m)
T^s:	is the slant tropospheric delay (m)
I_f^s:	is the slant ionosphere delay on frequency f (m)
b_r^s:	is the code bias (TGD) for pesudorange (m)
N_f^s:	is the phase ambiguity on frequency f (cycle)
λ_f:	is the wave length on frequency f (m/cycle)
ε:	is the measurement noise, including the multipath effect (m)

Other terms: Relativistic effect; PCO/PCV; Solid earth tides; Polar tides; Ocean loading; phase wind up etc.

GNSS无电离层组合PPP观测方程

$$\begin{cases} P_f^s = \rho_r^s + t_r + T^s + \frac{40.3}{f^2}I_r^s - b^{s,f} + b_{r,f} + \varepsilon_P \\ \Phi_f^s = \rho_r^s + t_r + T^s - \frac{40.3}{f^2}I_r^s - \lambda_f N_{r,f}^s + \varepsilon_\Phi \end{cases}$$

$$\begin{pmatrix} P_{LC}^s \\ \Phi_{LC}^s \end{pmatrix} = \begin{pmatrix} \frac{f_1^2}{f_1^2 - f_2^2} & \frac{-f_2^2}{f_1^2 - f_2^2} \end{pmatrix} \begin{pmatrix} P_1^s & \Phi_1^s \\ P_2^s & \Phi_2^s \end{pmatrix}$$

- 由双频伪距观测值

$$\begin{cases} P_1^s = \rho_r^s + t_r + T^s + \frac{40.3}{f_1^2}I_r^s + b_{r,1} + \varepsilon_P \\ P_2^s = \rho_r^s + t_r + T^s + \frac{40.3}{f_2^2}I_r^s + b_{r,2} + \varepsilon_P \end{cases}$$

$$\begin{aligned} P_{LC}^s &= \rho_r^s + t_r + T^s + \left(\frac{f_1^2}{f_1^2-f_2^2}b_{r,1} - \frac{f_2^2}{f_1^2-f_2^2}b_{r,2}\right) + \varepsilon_{P,LC} \\ &= \rho_r^s + t_r + T^s + b_{r,LC} + \varepsilon_{P,LC} \end{aligned}$$

- 由双频相位观测值

$$\begin{cases} \Phi_f^s = \rho_r^s + t_r + T^s - \frac{40.3}{f_1^2}I_r^s - \lambda_1 N_{r,1}^s + \varepsilon_\Phi \\ \Phi_f^s = \rho_r^s + t_r + T^s - \frac{40.3}{f_2^2}I_r^s - \lambda_2 N_{r,2}^s + \varepsilon_\Phi \end{cases}$$

$$\begin{aligned} \Phi_{LC}^s &= \rho_r^s + t_r + T^s - \lambda_1\left(\frac{f_1^2}{f_1^2-f_2^2}N_{r,1}^s - \frac{\lambda_2 f_2^2}{\lambda_1(f_1^2-f_2^2)}N_{r,2}^s\right) + \varepsilon_{\Phi,LC} \\ &= \rho_r^s + t_r + T^s - \lambda_1 N_{r,LC}^s + \varepsilon_{\Phi,LC} \end{aligned}$$

无电离层组合PPP观测方程

$$\begin{cases} P_{LC}^s = \rho_r^s + t_r + T^s + b_{r,LC} + \varepsilon_{P,LC} \\ \Phi_{LC}^s = \rho_r^s + t_r + T^s - \lambda_1 N_{r,LC}^s + \varepsilon_{\Phi,LC} \end{cases}$$

- □ 将观测方程在初始位置$\boldsymbol{r_0} = (x_0 \quad y_0 \quad z_0)$处按泰勒级数展开，保留一阶项

$$\begin{cases} P_{LC}^s = \rho_0^s - A_{X_r}^s \delta r_{GNSS} + t_r + T^s + b_{r,LC} + \varepsilon_{P,LC} \\ \Phi_{LC}^s = \rho_0^s - A_{X_r}^s \delta r_{GNSS} + t_r + T^s - \lambda_1 N_{r,LC}^s + \varepsilon_{\Phi,LC} \end{cases}$$

*Tips： $\rho = \{(\boldsymbol{x^s} - \boldsymbol{x_r})^2 + (\boldsymbol{y^s} - \boldsymbol{y_r})^2 + (\boldsymbol{z^s} - \boldsymbol{z_r})^2\}^{1/2} \approx \{(\boldsymbol{x^s} - \boldsymbol{x_r})^2 + (\boldsymbol{y^s} - \boldsymbol{y_r})^2 + (\boldsymbol{z^s} - \boldsymbol{z_r})^2\}^{1/2}\Big|_{r_0} + \frac{\partial\rho}{\partial x}\Big|_{r_0} dx + \frac{\partial\rho}{\partial y}\Big|_{r_0} dy + \frac{\partial\rho}{\partial z}\Big|_{r_0} dz$

$$= \rho_0 - \frac{x^s - x_0}{\rho_0}\Delta x - \frac{y^s - y_0}{\rho_0}\Delta y - \frac{z^s - z_0}{\rho_0}\Delta z = \rho_0 - \begin{pmatrix} \frac{x^s - x_0}{\rho_0} & \frac{y^s - y_0}{\rho_0} & \frac{z^s - z_0}{\rho_0} \end{pmatrix} (\Delta x \quad \Delta y \quad \Delta z)^{\mathrm{T}}$$

误差项	改正方式	文件ftp / 参考文献
精密星历 (轨道和钟差)	文件 sp3/clk	ftp://cddis.gsfc.nasa.gov/pub/gps/products/mgex/ ftp://igs.gnsswhu.cn/pub/gnss/products/mgex
PCO/PCV	文件 atx	ftp://garner.ucsd.edu/pub/gamit/tables/
伪距硬件延迟	文件 dcb	ftp://cddis.gsfc.nasa.gov/pub/gps/products/mgex/dcb
相对论效应	模型	IERS Conventions Centre 2010
相位缠绕	模型	Wu J, Hajj G A, Wu S, et al. Effects of antenna orientation on GPS carrier phase
固体潮、海潮、极潮	模型	IERS Conventions Centre 2010

无电离层组合PPP观测方程(续)

$$\begin{cases} P_{LC}^s = \rho_0^s - A_{X_r}^s \delta r_{GNSS} + t_r + T^s + b_{r,LC} + \varepsilon_{P,LC} \\ \Phi_{LC}^s = \rho_0^s - A_{X_r}^s \delta r_{GNSS} + t_r + T^s - \lambda_1 N_{r,LC}^s + \varepsilon_{\Phi,LC} \end{cases}$$

- 由先验对流层延迟模型得到对流层改正数

$$T_0^s = m_h^s T_h + m_w^s T_w$$

- 先验对流层湿延迟改正精度有限
  - 对流层湿延迟依赖于水汽含量等气象参数，变化复杂
  - GPT2w模型天顶对流层延迟精度约为4cm
- 将先验对流层改正数带入PPP方程，同时将天顶湿延迟改正数作为待估参数

$$\begin{cases} P_{LC}^s = \rho_0^s - A_{X_r}^s \delta r_{GNSS} + t_r + T_0^s + m_w^s \delta T_w + b_{r,LC} + \varepsilon_{P,LC} \\ \Phi_{LC}^s = \rho_r^s - A_{X_r}^s \delta r_{GNSS} + t_r + T_0^s + m_w^s \delta T_w - \lambda_1 N_{r,LC}^s + \varepsilon_{\Phi,LC} \end{cases}$$

- 误差方程

$$\begin{cases} V_{P_{LC}}^s = -A_{X_r}^s \delta r_{GNSS} + t_r + m_w^s \delta T_w + b_{r,LC} - (P_{LC}^s - \rho_0^s - T_0^s) \leftarrow l_{P_{LC}}^s \\ V_{\Phi_{LC}}^s = -A_{X_r}^s \delta r_{GNSS} + t_r + +m_w^s \delta T_w - \lambda_1 N_{r,LC}^s - (\Phi_{LC}^s - \rho_r^s - T_0^s) \leftarrow l_{\Phi_{LC}}^s \end{cases}$$

接收机钟差与伪距偏差

$$\begin{cases} V_{P_{LC}}^s = -A_{X_r}^s \delta r_{GNSS} + t_r + m_w^s \delta T_w + b_{r,LC} - l_{P_{LC}}^s \\ V_{\Phi_{LC}}^s = -A_{X_r}^s \delta r_{GNSS} + t_r + +m_w^s \delta T_w - \lambda_1 N_{r,LC}^s - l_{\Phi_{LC}}^s \end{cases}$$

$$A_{rGNSS}^s = \begin{pmatrix} \frac{x^s - x_{r0}}{\rho_0} & \frac{y^s - y_{r0}}{\rho_0} & \frac{z^s - z_{r0}}{\rho_0} \end{pmatrix}$$

- 假设观测到 m 颗卫星，即$s \in (1 \quad 2 \quad \cdots \quad m)$
- 则有矩阵形式的观测方程(仅考虑伪距时，SPP)

$$L_P = \begin{pmatrix} l_P^1 \\ l_P^2 \\ \vdots \\ l_P^m \end{pmatrix} = \begin{pmatrix} -A_{rGNSS}^1 & m_w^1 & 1 & 1 \\ -A_{rGNSS}^2 & m_w^2 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -A_{rGNSS}^m & m_w^m & 1 & 1 \end{pmatrix} \begin{pmatrix} \delta r_{GNSS} \\ \delta T_w \\ t_r \\ b_{r,LC} \end{pmatrix} = A_X X$$

- 设计矩阵A秩亏， $r = rank(A_X) = 5 < \mathrm{col}(A_X) = 6$，设矩阵$B$为

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- 容易验证$r = rank\begin{pmatrix} A_X \\ B \end{pmatrix} = 6 = \mathrm{col}\begin{pmatrix} A_X \\ B \end{pmatrix}$，即$R(A_X^T) + R(B^T) = R^6$，则有满秩系统

$$\begin{pmatrix} L_P \\ 0 \end{pmatrix} = \begin{pmatrix} A_X \\ B \end{pmatrix} X$$

- 此时接收机钟差满足

$$t_r := t_r + b_{r,LC}$$

无电离层组合PPP观测方程(续)

$$\begin{cases} V_{P_{LC}}^{s} = -A_{X_r}^{s}\delta r_{GNSS} + t_r + m_w^s dT_w - l_{P_{LC}}^{s} \\ V_{\Phi_{LC}}^{s} = -A_{X_r}^{s}\delta r_{GNSS} + t_r + m_w^s dT_w - \lambda_1 N_{r,LC}^{s} - l_{\Phi_{LC}}^{s} \end{cases}$$

$$t_r \equiv t_r + b_{r,LC}$$

- 矩阵形式的观测方程

$$L = \begin{pmatrix} L_P \\ L_\Phi \end{pmatrix} = \begin{pmatrix} A_X & u & Z \\ A_X & u & H_N \end{pmatrix} \begin{pmatrix} X \\ t_r \\ N_{LC} \end{pmatrix}$$

$$L_P = \begin{pmatrix} -A_{r_{GNSS}}^{1} & m_w^1 & 1 \\ -A_{r_{GNSS}}^{2} & m_w^2 & 1 \\ \vdots & \vdots & \vdots \\ -A_{r_{GNSS}}^{m} & m_w^m & 1 \end{pmatrix} \begin{pmatrix} \delta r_{GNSS} \\ \delta T_w \\ t_r \end{pmatrix}$$

- 其中

$$\begin{cases} X = \begin{pmatrix} \delta r_{GNSS} \\ \delta T_w \end{pmatrix} \\ N_{LC} = \begin{pmatrix} N_{LC}^1 & N_{LC}^2 & \cdots & N_{LC}^m \end{pmatrix}^T \end{cases}$$

$$N_{r,LC}^{s} \equiv N_{r,LC}^{s} + \frac{b_{r,LC}}{\lambda_1}$$

$$A_X = \begin{pmatrix} -A_{X_r}^{1} & m_w^1 \\ -A_{X_r}^{2} & m_w^2 \\ \vdots & \vdots \\ -A_{X_r}^{m} & m_w^m \end{pmatrix} \qquad H_N = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_1 \end{pmatrix} = \lambda_1 U$$

*Tips： $Z = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ $U = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$ $u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

无电离层组合PPP随机模型

$$\begin{pmatrix} P_{LC}^S \\ \Phi_{LC}^S \end{pmatrix} = \begin{pmatrix} \frac{f_1^2}{f_1^2 - f_2^2} & \frac{-f_2^2}{f_1^2 - f_2^2} \end{pmatrix} \begin{pmatrix} P_1^S & \Phi_1^S \\ P_2^S & \Phi_2^S \end{pmatrix}$$

$$\begin{cases} D_{\varepsilon_{P_{LC}}} = 8.87 D_{\varepsilon_P} \\ D_{\varepsilon_{\Phi_{LC}}} = 8.87 D_{\varepsilon_\Phi} \end{cases}$$

- 接收机相位观测量精度比伪距观测量精度高两个数量级，不同卫星观测值之间独立同分布IID
  - 假设（非差非组合）相位/伪距中误差分别为σ_0、$100\sigma_0$
  - 则无电离层组合相位/伪距中误差分别为$3\sigma_0$、$300\sigma_0$
  - 则观测向量L方差-协方差矩阵D_L为

$$D_L = 8.87\sigma_0^2 \begin{pmatrix} 100^2 I & Z \\ Z & I \end{pmatrix}$$

- 高度角加权因子γ，$\sigma_0^E = \gamma\sigma_0$，其中$\gamma = \begin{cases} 1 & ; E \geq 30^\circ \\ 1/2sin(E) & ; E < 30^\circ \end{cases}$
- 相位中误差取值σ_0
  - 测量型接收机σ_0 =0.003m
  - 导航型接收机σ_0 =0.03m
  - M估计

无电离层组合PPP状态方程

$$\boldsymbol{L}=\begin{pmatrix}\boldsymbol{L_P}\\ \boldsymbol{L_\Phi}\end{pmatrix}=\begin{pmatrix}\boldsymbol{A_X} & \boldsymbol{u} & \boldsymbol{Z}\\ \boldsymbol{A_X} & \boldsymbol{u} & \boldsymbol{H_N}\end{pmatrix}\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{t_r}\\ \boldsymbol{N_{LC}}\end{pmatrix}$$

$$\begin{cases}\boldsymbol{X}=\begin{pmatrix}\boldsymbol{dr_{GNSS}}\\ \boldsymbol{dT_w}\end{pmatrix}\\ \boldsymbol{N_{LC}}=\begin{pmatrix}\boldsymbol{N_{LC}^1} & \boldsymbol{N_{LC}^2} & \cdots & \boldsymbol{N_{LC}^m}\end{pmatrix}^{\boldsymbol{T}}\end{cases}$$

$$\boldsymbol{\Phi_{k+1,k}}=\boldsymbol{exp}\left(\int_{t_j}^{t_{j+1}}F(t)\boldsymbol{dt}\right)$$

$$\boldsymbol{w_j}=\int_{t_j}^{t_{j+1}}\boldsymbol{e^{-(t_{j+1}-\xi)/\tau}\omega(\xi)d\xi}$$

- 离散卡尔曼滤波状态方程

$$\dot{x}(t)=F(t)x(t)+G(t)\omega(t)$$

$$x_{j+1}=\Phi_{j+1,j}x_j+w_j$$

- 位置误差δr_{GNSS}、接收机钟差t_r一般建模为白噪声

$$\begin{cases}p_{j+1}=w_j\\ E(w_j)=0\\ E(w_j^2)=D_{w_j}\end{cases}$$

- 对流层湿延迟残差δT_w一般建模为随机游走噪声

$$\begin{cases}p_{j+1}=p_j+w_j\\ E(w_j)=0\\ E(w_j^2)=D_{w_j}\end{cases}$$

- 浮点模糊度$\boldsymbol{N_{LC}}$为常数

无电离层组合PPP

$$\begin{cases} V^s_{P_{LC}} = -A^s_{X_r}\delta r_{GNSS} + t_r + m^s_w\delta T_w + b_{r,LC} - l^s_{P_{LC}} \\ V^s_{\Phi_{LC}} = -A^s_{X_r}\delta r_{GNSS} + t_r + +m^s_w\delta T_w - \lambda_1 N^s_{r,LC} - l^s_{\Phi_{LC}} \end{cases}$$

- □ 观测方程

$$L = \begin{pmatrix} L_P \\ L_\Phi \end{pmatrix} = \begin{pmatrix} A_X & u & Z \\ A_X & u & H_N \end{pmatrix} \begin{pmatrix} X \\ t_r \\ N_{LC} \end{pmatrix}$$

- □ 随机模型

$$D_L = 8.87\sigma_0^2 \begin{pmatrix} 100^2 I & Z \\ Z & I \end{pmatrix} \qquad \sigma_0^E = \gamma\sigma_0 \ , \ \gamma = \begin{cases} 1 & ; E \geq 30^\circ \\ 1/2sin(E) & ; E < 30^\circ \end{cases}$$

- □ 状态方程

$$\begin{pmatrix} dr_{GNSS} \\ dT_w \\ t_r \\ N_{LC} \end{pmatrix}_{j+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta r_{GNSS} \\ \delta T_w \\ t_r \\ N_{LC} \end{pmatrix}_j + \begin{pmatrix} w_{\delta r} \\ w_{\delta T_w} \\ w_{t_r} \\ 0 \end{pmatrix}_j$$

惯导误差方程

- 惯导仪器误差模型
  - 陀螺测量模型：$\tilde{\boldsymbol{\omega}}_{ib}^{b} = \boldsymbol{\omega}_{ib}^{b} + \boldsymbol{b}_g + \boldsymbol{s}_g\boldsymbol{\omega}_{ib}^{b} + \boldsymbol{N}_g\boldsymbol{\omega}_{ib}^{b} + \boldsymbol{\varepsilon}_{\omega}$
  - 加速度计测量模型：$\tilde{\boldsymbol{f}}^{b} = \boldsymbol{f}^{b} + \boldsymbol{b}_a + \boldsymbol{s}_a\boldsymbol{f}^{b} + \boldsymbol{N}_g\boldsymbol{f}^{b} + \boldsymbol{\varepsilon}_f$
    - $\boldsymbol{b}$：陀螺仪和加速度计的零偏
    - $\boldsymbol{s}$：陀螺仪和加速度计的比例因子误差
    - $\boldsymbol{N}$：陀螺仪和加速度计的交轴耦合误差

$$\boldsymbol{s} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$$

$$\boldsymbol{N} = \begin{pmatrix} 0 & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & 0 & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & 0 \end{pmatrix}$$

- 由测量模型得到仪器误差模型（忽略交轴耦合误差）：

$$\begin{cases} \delta\boldsymbol{\omega}_{ib}^{b} = \boldsymbol{b}_g + \boldsymbol{s}_g\boldsymbol{\omega}_{ib}^{b} \\ \delta\boldsymbol{f}_b = \boldsymbol{b}_a + \boldsymbol{s}_a\boldsymbol{f}^{b} \end{cases}$$

- e系下，惯导状态微分方程

$$\begin{cases} \dot{\boldsymbol{r}}_{eb}^{e} = \boldsymbol{v}_{eb}^{e} \\ \dot{\boldsymbol{v}}_{eb}^{e} = \boldsymbol{C}_b^e(\boldsymbol{f}_b) - 2\boldsymbol{\omega}_{ie}^{e}\times\boldsymbol{v}_{eb}^{e} + \boldsymbol{g}^e \\ \dot{\boldsymbol{C}}_b^e = \boldsymbol{C}_b^e[\boldsymbol{\omega}_{eb}^{b}\times] \end{cases} \Leftarrow$$

$$\left.\frac{d\boldsymbol{v}_e}{dt}\right|_R = \boldsymbol{f} - (\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{iR})\times\boldsymbol{v}_e + \boldsymbol{g}$$

$$\dot{\boldsymbol{C}}_b^R = \boldsymbol{C}_b^R[\boldsymbol{\omega}_{Rb}^{b}\times]$$

R为任意坐标系
g为地球重力加速度

误差扰动

$$\begin{cases}\dot{\mathbf{r}}_{eb}^{e}=\mathbf{v}_{eb}^{e}\\ \dot{\mathbf{v}}_{eb}^{e}=\mathbf{C}_{b}^{e}(\mathbf{f}_{b})-2\boldsymbol{\omega}_{ie}^{e}\times\mathbf{v}_{eb}^{e}+\mathbf{g}^{e}\\ \dot{\mathbf{C}}_{b}^{e}=\mathbf{C}_{b}^{e}[\boldsymbol{\omega}_{eb}^{b}\times]\end{cases}$$

$$\begin{cases}\delta\boldsymbol{\omega}_{ib}^{b}=\boldsymbol{b}_{g}+\boldsymbol{s}_{g}\boldsymbol{\omega}_{ib}^{b}\\ \delta\boldsymbol{f}_{b}=\boldsymbol{b}_{a}+\boldsymbol{s}_{a}\boldsymbol{f}^{b}\end{cases}$$

$$\boldsymbol{C}_{b}^{e'}=(\boldsymbol{I}-\boldsymbol{\phi}\times)\boldsymbol{C}_{b}^{e}$$

$$\delta\boldsymbol{C}_{b}^{e}=-\boldsymbol{\phi}\times\boldsymbol{C}_{b}^{e}$$

$$\delta\dot{\boldsymbol{C}}_{b}^{e}=-\dot{\boldsymbol{\phi}}\times\boldsymbol{C}_{b}^{e}-\boldsymbol{\phi}\times\dot{\boldsymbol{C}}_{b}^{e}$$

- 误差扰动等效于围绕真值进行泰勒展开，取至一阶项得惯导状态误差微分方程

$$\begin{cases}\delta\dot{r}_{eb}^{e}=\delta v_{eb}^{e}\\ \delta\dot{v}_{eb}^{e}=\delta C_{b}^{e}f_{b}+C_{b}^{e}\delta f_{b}-2\omega_{ie}^{e}\times\delta v_{eb}^{e}+\delta g^{e}\\ \delta\dot{C}_{b}^{e}=\delta C_{b}^{e}\Omega_{eb}^{b}+C_{b}^{e}\delta\Omega_{eb}^{b}\end{cases}$$

$\delta\mathbf{v}_{eb}^{e}$：速度误差

$\delta\mathbf{C}_{b}^{e}$、$\boldsymbol{\phi}$：姿态误差

$\delta\mathbf{f}_{b}$：加速度计误差

$\delta\boldsymbol{\omega}_{ib}^{b}$：陀螺误差

$\delta\mathbf{g}^{e}$：重力误差项，考虑计算效率，可忽略

- 将仪器误差模型带入惯导误差方程，并采用$\boldsymbol{\phi}$角失准角误差模型描述姿态误差状态

$$\begin{cases}\delta\dot{r}_{eb}^{e}=\delta v_{eb}^{e}\\ \delta\dot{v}_{eb}^{e}=[C_{b}^{e}f_{b}\times]\phi+C_{b}^{e}(b_{a}+s_{a}f^{b})-2\omega_{ie}^{e}\times\delta v_{eb}^{e}\\ \dot{\phi}=-C_{b}^{e}(b_{g}+s_{g}\omega_{ib}^{b})-\omega_{ie}^{e}\times\phi\end{cases}$$

- 进一步，零偏、比例因子误差建模为一阶高斯-马尔科夫过程:

$$\begin{aligned}\dot{\boldsymbol{b}}_{g}&=-\boldsymbol{b}_{g}/T_{bg}+\varepsilon_{bg}\\ \dot{\boldsymbol{b}}_{a}&=-\boldsymbol{b}_{a}/T_{ba}+\varepsilon_{ba}\\ \dot{\boldsymbol{s}}_{g}&=-\boldsymbol{s}_{g}/T_{sg}+\varepsilon_{sg}\\ \dot{\boldsymbol{s}}_{a}&=-\boldsymbol{s}_{a}/T_{sa}+\varepsilon_{sa}\end{aligned}$$

惯导状态误差微分方程

$$\begin{cases}\delta\dot{r}_{eb}^{e}=\delta v_{eb}^{e}\\ \delta\dot{v}_{eb}^{e}=[C_b^e\boldsymbol{f}_b\times]\boldsymbol{f}_b\times\phi+C_b^e(b_a+s_af^b)-2\omega_{ie}^e\times\delta v_{eb}^e\\ \dot{\phi}=-C_b^e(b_g+s_g\omega_{ib}^b)-\omega_{ie}^e\times\phi\end{cases}$$

$$\dot{\boldsymbol{b}}_g=-\boldsymbol{b}_g/T_{bg}+\varepsilon_{bg}$$
$$\dot{\boldsymbol{b}}_a=-\boldsymbol{b}_a/T_{ba}+\varepsilon_{ba}$$
$$\dot{\boldsymbol{s}}_g=-\boldsymbol{s}_g/T_{sg}+\varepsilon_{sg}$$
$$\dot{\boldsymbol{s}}_a=-\boldsymbol{s}_a/T_{sa}+\varepsilon_{sa}$$

- 惯导状态误差微分方程：

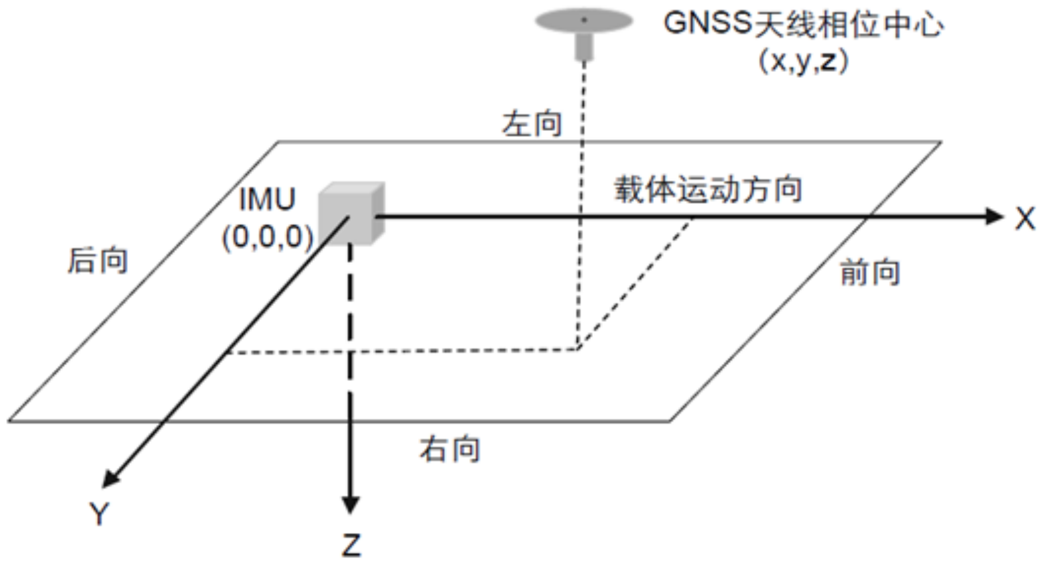
$$\begin{cases}\delta\dot{r}_{eb}^{e}=\delta v_{eb}^{e}\\ \delta\dot{v}_{eb}^{e}=[C_b^ef_b\times]\phi+C_b^e(b_a+s_af^b)-2\omega_{ie}^e\times\delta v_{eb}^e\\ \dot{\phi}=-C_b^e(b_g+s_g\omega_{ib}^b)-\omega_{ie}^e\times\phi\\ \dot{b}_g=-b_g/T_{bg}+\varepsilon_{bg}\\ \dot{b}_a=-b_a/T_{ba}+\varepsilon_{ba}\\ \dot{s}_g=-s_g/T_{sg}+\varepsilon_{sg}\\ \dot{s}_a=-s_a/T_{sa}+\varepsilon_{sa}\end{cases}$$

- e系下矩阵形式状态误差微分方程， $\dot{X}(t)=F(t)X(t)+G(t)W(t)$

$$\boldsymbol{F}=\begin{pmatrix}0 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0\\ 0 & -2\boldsymbol{\omega}_{ie}^e\times & [C_b^e\boldsymbol{f}_b\times] & 0 & C_b^e & 0 & C_b^ef^b\\ 0 & 0 & -\boldsymbol{\omega}_{ie}^e\times & -C_b^e & 0 & -C_b^e\omega_{ib}^b & 0\\ 0 & 0 & 0 & -\boldsymbol{I}/T_{bg} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{ba} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{sg} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{sa}\end{pmatrix}$$

杆臂

- GNSS状态量$x_{GNSS}=\left(\delta r_{GNSS}{}^{T}\quad \delta T_{w}\quad t_{r}\quad N_{LC}{}^{T}\right)^{T}$
- IMU状态量$x_{IMU}=\left(\delta r_{IMU}{}^{T}\quad \delta v_{INS}{}^{T}\quad \phi^{T}\quad b_{g}{}^{T}\quad b_{a}{}^{T}\quad s_{g}{}^{T}\quad s_{a}{}^{T}\right)^{T}$
- 天线相位中心$\boldsymbol{x}_{GNSS}^{e}$与载体坐标系原点$\boldsymbol{x}_{IMU}^{e}$物理上一般不重合
  - 组合导航解算时需进行杆臂效应改正



$$\boldsymbol{r}_{GNSS}^{e}=\boldsymbol{r}_{IMU}^{e}+\boldsymbol{C}_{b}^{e}\boldsymbol{l}^{b}\times\boldsymbol{\phi}$$ 考虑姿态误差

$$\boldsymbol{v}_{GNSS}^{e}=\boldsymbol{v}_{IMU}^{e}-[\boldsymbol{\omega}_{ie}^{e}\times]\boldsymbol{C}_{b}^{e}\boldsymbol{l}^{b}-\boldsymbol{C}_{b}^{e}[\boldsymbol{l}^{b}\times]\boldsymbol{\omega}_{ib}^{b}$$

扰动分析

$$r^e_{GNSS} = r^e_{IMU} + C^e_b l^b \times \phi$$

$$r^e_{GNSS} = r^e_{IMU} - [\omega^e_{ie}\times]C^e_b l^b - C^e_b[l^b\times]\omega^b_{ib}$$

- 位置

$$\begin{aligned}
\delta r^e_{GNSS} &= r^e_{GNSS} - \tilde{r}^e_{GNSS} \\
&= r^e_{GNSS} - (\tilde{r}^e_{IMU} + \tilde{C}^e_b l^b) \\
&= r^e_{GNSS} - (r^e_{IMU} + \delta r^e_{IMU} + (I - \phi\times)C^e_b l^b) \\
&= -\delta r^e_{IMU} - C^e_b l^b \times \phi + (r^e_{GNSS} - r^e_{IMU} - C^e_b l^b) \\
&= -\delta r^e_{IMU} - C^e_b l^b \times \phi
\end{aligned}$$

- 速度（GNSS使用多普勒观测值测速时才考虑）

$$\begin{aligned}
\delta v^e_{GNSS} &= v^e_{GNSS} - \tilde{v}^e_{GNSS} \\
&= v^e_{GNSS} - (\tilde{v}^e_{IMU} - [\omega^e_{ie}\times]\tilde{C}^e_b l^b - \tilde{C}^e_b[l^b\times]\tilde{\omega}^b_{ib}) \\
&= v^e_{GNSS} - \left(v^e_{IMU} + \delta v^e_{IMU} - [\omega^e_{ie}\times](\mathrm{I} - \phi\times)C^e_b l^b - (\mathrm{I} - \phi\times)[l^b\times](\omega^e_{ib} + \delta\omega^e_{ib})\right) \\
&= -(\delta v^e_{IMU} - ([\omega^e_{ie}\times]\, C^e_b\, l^b + C^e_b[l^b\times]\, \omega^e_{ib}))\times\phi - C^e_b[l^b\times]\delta\omega^e_{ib} + \\
&\quad (v^e_{GNSS} - (v^e_{IMU} - [\omega^e_{ie}\times]\, C^e_b\, l^b - C^e_b[l^b\times]\delta\omega^e_{ib})) \\
&= (\delta v^e_{IMU} + \tilde{C}^e_b[l^b\times]\tilde{\omega}^b_{ib} + ([\omega^e_{ie}\times]\, \tilde{C}^e_b l^b + \tilde{C}^e_b[l^b\times]\, \omega^e_{ib})\times\phi
\end{aligned}$$

紧组合观测方程

- 将GNSS和IMU之间的位置约束关系代入到PPP观测方程可得到紧组合观测方程

$$\delta r_{GNSS}^{e} = -\delta r_{IMU}^{e} - C_b^e l^b \times \phi$$

$$+$$

$$\begin{cases} V_{P_{LC}}^{s} = -A_{X_r}^{s}\delta r_{GNSS} + t_r + m_w^s dT_w - l_{P_{LC}}^{s} \\ V_{\Phi_{LC}}^{s} = -A_{X_r}^{s}\delta r_{GNSS} + t_r + m_w^s dT_w - \lambda_1 N_{r,LC}^{s} - l_{\Phi_{LC}}^{s} \end{cases}$$

$$\begin{cases} V_{P_{LC}}^{s} = A_{X_r}^{s}\delta r_{IMU}^{e} + A_{X_r}^{s} C_b^e l^b \times \phi + t_r + m_w^s dT_w - l_{P_{LC}}^{s} \\ V_{\Phi_{LC}}^{s} = A_{X_r}^{s}\delta r_{IMU}^{e} + A_{X_r}^{s} C_b^e l^b \times \phi + t_r + m_w^s dT_w - \lambda_1 N_{r,LC}^{s} - l_{\Phi_{LC}}^{s} \end{cases}$$

紧组合状态方程

$$x_{COM}=\left(\delta r_{IMU}{}^T \quad \delta v_{INS}{}^T \quad \phi^T\right)^T$$
$$x_{IMU}=\left(b_g{}^T \quad b_a{}^T \quad s_g{}^T \quad s_a{}^T\right)^T$$
$$x_{GNSS}=\left(\delta T_w \quad t_r \quad N_{LC}{}^T\right)^T$$

$$\begin{pmatrix}\dot{\boldsymbol{x}}_{COM}\\ \dot{\boldsymbol{x}}_{IMU}\\ \dot{\boldsymbol{x}}_{GNSS}\end{pmatrix}=\begin{pmatrix}\boldsymbol{F}_C & \boldsymbol{F}_{C,I} & 0_{9*(2+m)}\\ \boldsymbol{0} & \boldsymbol{F}_I & 0_{9*(2+m)}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{F}_G\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_{COM}\\ \boldsymbol{x}_{IMU}\\ \boldsymbol{x}_{GNSS}\end{pmatrix}+\begin{pmatrix}\boldsymbol{\omega}_{COM}\\ \boldsymbol{\omega}_{IMU}\\ \boldsymbol{\omega}_{GNSS}\end{pmatrix}$$

$$\dot{\boldsymbol{x}}_{TC}=\boldsymbol{F}\cdot\boldsymbol{x}_{TC}+\boldsymbol{G}\cdot\boldsymbol{w}$$

$$\boldsymbol{F}=\begin{pmatrix}
0 & I_{3*3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & -2\omega_{ie}^e\times & C_b^e f_b\times & 0 & C_b^e & 0 & C_b^e f^b & 0 & 0 & 0 & \cdots\\
0 & 0 & -\omega_{ie}^e\times & -C_b^e & 0 & -C_b^e\omega_{ib}^b & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & -\boldsymbol{I}/T_{bg} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{ba} & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{sg} & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{sa} & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{\delta T_w} & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{I}/T_{t_r} & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

当相关时间$T\to\infty$时，一阶高斯-马尔科夫过程即为随机游走过程
当相关时间$T\to 0$时，一阶高斯-马尔科夫过程即为白噪声过程

离散化后的状态方程

$$x_{COM} = (\delta r_{IMU}{}^T \quad \delta v_{INS}{}^T \quad \phi^T)^T$$
$$x_{IMU} = (b_g{}^T \quad b_a{}^T \quad s_g{}^T \quad s_a{}^T)^T$$
$$x_{GNSS} = (\delta T_w \quad t_r \quad N_{LC}{}^T)^T$$

$$\begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_{j+1} = \begin{pmatrix} \boldsymbol{\Phi}_C & \boldsymbol{\Phi}_{C,I} & \mathbf{0}_{9*(2+m)} \\ \mathbf{0} & \boldsymbol{\Phi}_I & \mathbf{0}_{9*(2+m)} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi}_G \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_{j} + \begin{pmatrix} \boldsymbol{w}_{COM} \\ \boldsymbol{w}_{IMU} \\ \boldsymbol{w}_{GNSS} \end{pmatrix}$$

$$\boldsymbol{\Phi} = \begin{pmatrix}
I_{3*3} & I_{3*3}\Delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & I_{3*3} - 2(\omega_{ie}^e \times)\Delta t & C_b^e(f_b \times)\Delta t & 0 & C_b^e \Delta t & 0 & C_b^e f_b \Delta t & 0 & 0 & 0 & \cdots \\
0 & 0 & I_{3*3} - (\omega_{ie}^e \times)\Delta t & -C_b^e \Delta t & 0 & -C_b^e \omega_{ib}^b \Delta t & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & I_{3*3} - \frac{1}{T_{bg}}\Delta t & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & I_{3*3} - \frac{1}{T_{ba}}\Delta t & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & I_{3*3} - \frac{1}{T_{sg}}\Delta t & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & I_{3*3} - \frac{1}{T_{sa}}\Delta t & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

PPP/INS紧组合滤波

- 状态方程（时间更新）

$$\begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_{j+1} = \begin{pmatrix} \boldsymbol{\Phi}_C & \boldsymbol{\Phi}_{C,I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Phi}_I & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Phi}_G \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_j + \begin{pmatrix} \boldsymbol{w}_{COM} \\ \boldsymbol{w}_{IMU} \\ \boldsymbol{w}_{GNSS} \end{pmatrix}$$

- 观测方程（测量更新）

$$\boldsymbol{L}_{j+1} = \begin{pmatrix} \boldsymbol{L}_P \\ \boldsymbol{L}_{\Phi} \end{pmatrix}_{j+1} = \begin{pmatrix} \boldsymbol{A}_{COM} & \boldsymbol{0} & \boldsymbol{A}_P \\ \boldsymbol{A}_{COM} & \boldsymbol{0} & \boldsymbol{A}_{\Phi} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_j + \begin{pmatrix} \varepsilon_{P,LC} \\ \varepsilon_{\boldsymbol{\Phi},LC} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} A^s_{X_r} & \boldsymbol{0} & A^s_{X_r} C^e_b l^b & \boldsymbol{0} & \boldsymbol{m}^s_w & \boldsymbol{u} & \boldsymbol{0} \\ A^s_{X_r} & \boldsymbol{0} & A^s_{X_r} C^e_b l^b & \boldsymbol{0} & \boldsymbol{m}^s_w & \boldsymbol{u} & \boldsymbol{\lambda_1 I} \end{pmatrix}$$

- 联合状态方程与观测方程

$$\begin{pmatrix} \boldsymbol{v}_{COM} \\ \boldsymbol{v}_{IMU} \\ \boldsymbol{v}_{GNSS} \\ \varepsilon_{P,LC} \\ \varepsilon_{\boldsymbol{\Phi},LC} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}_C & \boldsymbol{\Phi}_{C,I} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Phi}_I & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Phi}_G & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{COM} & \boldsymbol{0} & \boldsymbol{A}_P \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{COM} & \boldsymbol{0} & \boldsymbol{A}_{\Phi} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_j \\ \begin{pmatrix} \boldsymbol{x}_{COM} \\ \boldsymbol{x}_{IMU} \\ \boldsymbol{x}_{GNSS} \end{pmatrix}_{j+1} \end{pmatrix} - \begin{pmatrix} E(\boldsymbol{w}_{COM}) \\ E(\boldsymbol{w}_{IMU}) \\ E(\boldsymbol{w}_{GNSS}) \\ \boldsymbol{L}_P \\ \boldsymbol{L}_{\Phi} \end{pmatrix}$$

PPP/INS紧组合滤波（续）

$$\begin{pmatrix} v_{COM} \\ v_{IMU} \\ v_{GNSS} \\ \varepsilon_{P,LC} \\ \varepsilon_{\Phi,LC} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}_C & \boldsymbol{\Phi}_{C,I} & 0 & -\boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{\Phi}_I & 0 & 0 & -\boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{\Phi}_G & 0 & 0 & -\boldsymbol{I} \\ 0 & 0 & 0 & \boldsymbol{A}_{COM} & 0 & \boldsymbol{A}_P \\ 0 & 0 & 0 & \boldsymbol{A}_{COM} & 0 & \boldsymbol{A}_{\Phi} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} x_{COM} \\ x_{IMU} \\ x_{GNSS} \end{pmatrix}_j \\ \begin{pmatrix} x_{COM} \\ x_{IMU} \\ x_{GNSS} \end{pmatrix}_{j+1} \end{pmatrix} - \begin{pmatrix} E(w_{COM}) \\ E(w_{IMU}) \\ E(w_{GNSS}) \\ \boldsymbol{L}_P \\ \boldsymbol{L}_{\Phi} \end{pmatrix}$$

- 法方程矩阵$\boldsymbol{N} = \boldsymbol{B}^T\boldsymbol{B}$（假设$\boldsymbol{P} = \boldsymbol{I}$）

$$\boldsymbol{N} = \begin{pmatrix} \boldsymbol{\Phi}_C{}^T & 0 & 0 & 0 & 0 \\ \boldsymbol{\Phi}_{C,I}{}^T & \boldsymbol{\Phi}_I{}^T & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\Phi}_G{}^T & 0 & 0 \\ -\boldsymbol{I} & 0 & 0 & \boldsymbol{A}_{COM}{}^T & \boldsymbol{A}_{COM}{}^T \\ 0 & -\boldsymbol{I} & 0 & 0 & 0 \\ 0 & 0 & -\boldsymbol{I} & \boldsymbol{A}_P{}^T & \boldsymbol{A}_{\Phi}{}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\Phi}_C & \boldsymbol{\Phi}_{C,I} & 0 & -\boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{\Phi}_I & 0 & 0 & -\boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{\Phi}_G & 0 & 0 & -\boldsymbol{I} \\ 0 & 0 & 0 & \boldsymbol{A}_{COM} & 0 & \boldsymbol{A}_P \\ 0 & 0 & 0 & \boldsymbol{A}_{COM} & 0 & \boldsymbol{A}_{\Phi} \end{pmatrix}$$

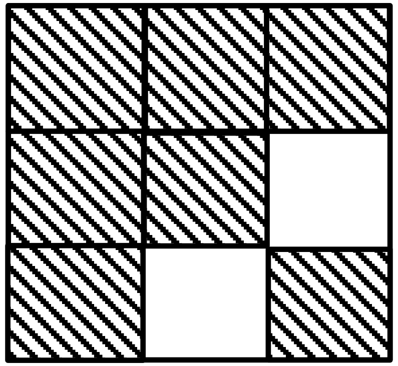
$$\boldsymbol{N} = \begin{pmatrix} \boldsymbol{\Phi}_C{}^T\boldsymbol{\Phi}_C & \boldsymbol{\Phi}_C{}^T\boldsymbol{\Phi}_{C,I} & 0 & -\boldsymbol{\Phi}_C{}^T & 0 & 0 \\ \boldsymbol{\Phi}_{C,I}{}^T\boldsymbol{\Phi}_C & \boldsymbol{\Phi}_{C,I}{}^T\boldsymbol{\Phi}_{C,I} + \boldsymbol{\Phi}_I{}^T\boldsymbol{\Phi}_I & 0 & -\boldsymbol{\Phi}_{C,I}{}^T & -\boldsymbol{\Phi}_I{}^T & 0 \\ 0 & 0 & \boldsymbol{\Phi}_G{}^T\boldsymbol{\Phi}_G & 0 & 0 & -\boldsymbol{\Phi}_G{}^T \\ \boldsymbol{\Phi}_C & -\boldsymbol{\Phi}_{C,I} & 0 & \boldsymbol{I} + 2\boldsymbol{A}_{COM}{}^T\boldsymbol{A}_{COM} & 0 & \boldsymbol{A}_{COM}{}^T(\boldsymbol{A}_P + \boldsymbol{A}_{\Phi}) \\ 0 & -\boldsymbol{\Phi}_I & 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & -\boldsymbol{\Phi}_G & (\boldsymbol{A}_P{}^T + \boldsymbol{A}_{\Phi}{}^T)\boldsymbol{A}_{COM} & 0 & \boldsymbol{I} + \boldsymbol{A}_P{}^T\boldsymbol{A}_P + \boldsymbol{A}_{\Phi}{}^T\boldsymbol{A}_{\Phi} \end{pmatrix}$$

- 状态量估值协因数阵$\boldsymbol{Q} = \boldsymbol{N}^{-1} = \dfrac{\boldsymbol{N}^*}{|\boldsymbol{N}|}$
  - $\boldsymbol{Q}_{I,G} \neq \boldsymbol{0}$

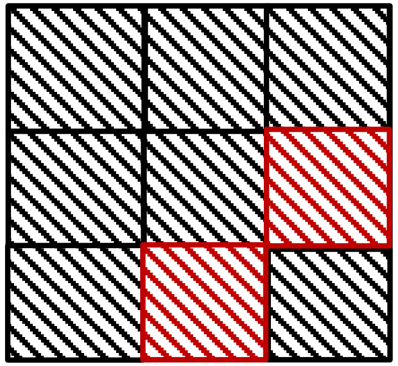
“一步解” VS “两步解”

$$\begin{pmatrix} x_a \\ x_{b|a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Q_{ba}Q_a^{-1} & 1 \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix} \quad Q' = \begin{pmatrix} Q_a & 0 \\ 0 & Q_b - Q_{ba}Q_a^{-1}Q_{ab} \end{pmatrix}$$

- $$\begin{cases} x_{COM} = \left(\delta r_{IMU}{}^T \quad \delta v_{INS}{}^T \quad \phi^T\right)^T \\ x_{IMU} = \left(b_g{}^T \quad b_a{}^T \quad s_g{}^T \quad s_a{}^T\right)^T \\ x_{GNSS} = \left(\delta T_w \quad t_r \quad N_{LC}{}^T\right)^T \end{cases}$$
- 法方程矩阵$\boldsymbol{N}$

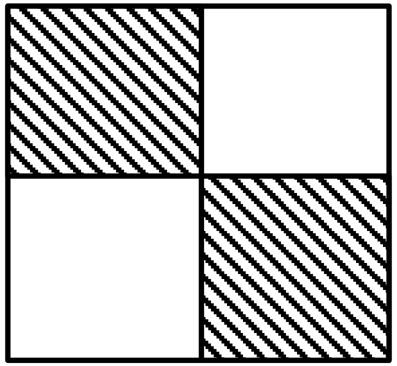


- 协因数阵 $\boldsymbol{Q}$

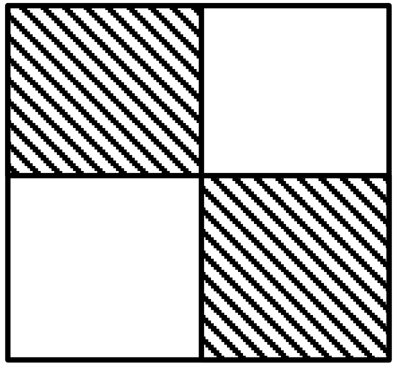


- 算法复杂度： $\boldsymbol{O}\left((\boldsymbol{n}_c + \boldsymbol{n}_I + \boldsymbol{n}_G)^3\right)$

- $$\begin{cases} x_{IMU} = \left(\delta r_{IMU}{}^T \quad \delta v_{INS}{}^T \quad \phi^T \quad b_g{}^T \quad b_a{}^T \quad s_g{}^T \quad s_a{}^T\right)^T \\ x_{GNSS} = \left(\delta r_{GNSS}{}^T \quad \delta T_w \quad t_r \quad N_{LC}{}^T\right)^T \end{cases}$$
- 法方程矩阵$\boldsymbol{N}$

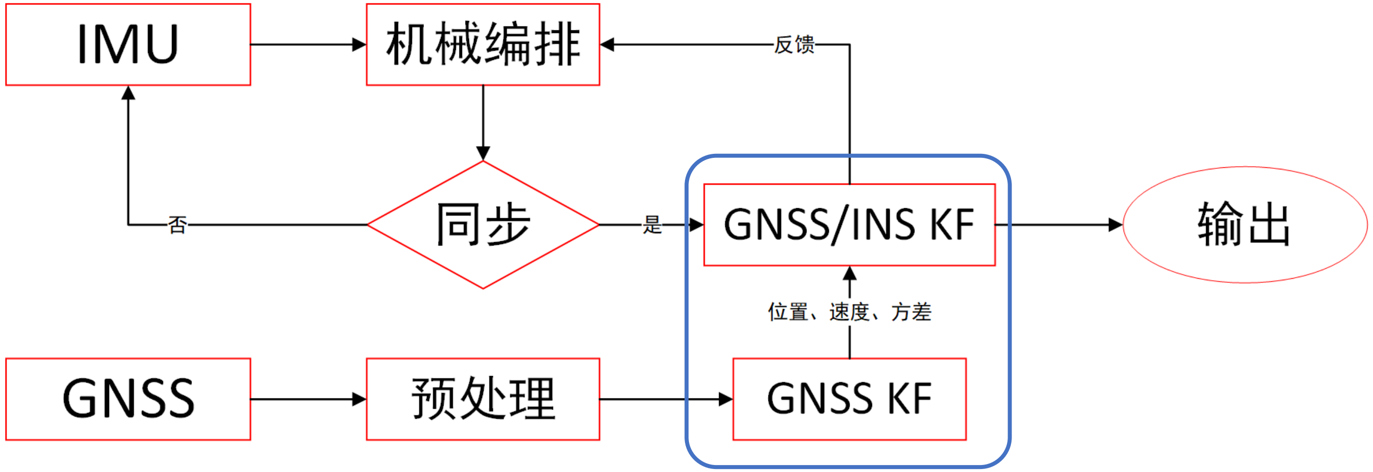


- 协因数阵 $\boldsymbol{Q}$

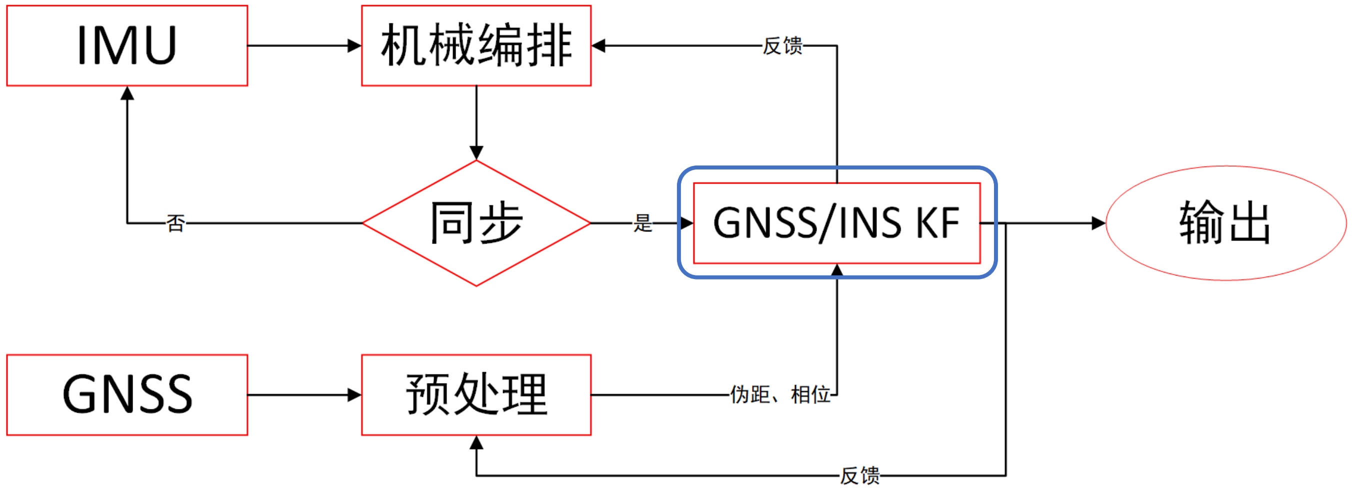


- 算法复杂度： $\boldsymbol{O}\left(\boldsymbol{n}_I{}^3 + \boldsymbol{n}_G{}^3\right)$

GNSS/INS组合



松组合



紧组合

谢 谢！

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